

# Testing for Monotone Equilibrium Strategies in Games of Incomplete Information<sup>1</sup>

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## Abstract

This paper develops a unified framework for testing monotonicity of Bayesian Nash equilibrium strategies in unobserved types in games of incomplete information. We show that, under symmetric independent private types, monotonicity of differentiable equilibrium strategies is equivalent to monotonicity of a quasi-inverse strategy identified from observed actions. This allows the problem to be reformulated as testing a countable set of moment inequalities involving unconditional expectations. We propose a Cramér–von Mises–type statistic with bootstrap critical values. The method accommodates covariates and game heterogeneity. Monte Carlo simulations demonstrate finite-sample performance, and an application to procurement auctions illustrates cartel detection.

*Keywords:* Monotone equilibrium strategies, Bayesian games, Moment inequalities, Rationalization, Testing for monotonicity, Game heterogeneity

*JEL codes:* C12, C14, L70

# 1 Introduction

In games of incomplete information, or Bayesian games, players' strategies are functions mapping their private types such as signals, beliefs, valuations, or costs, to actions. A Bayesian Nash equilibrium (BNE) is a profile of such strategy functions where each player's strategy is a best response to others, given beliefs about their types. Monotonicity of equilibrium strategies, which means that a player's action is monotone with their type, is one of the most empirically informative predictions in games of incomplete information: higher valuations, lower costs, or stronger signals lead to more aggressive actions. This "sorting" property underlies how economists interpret actions as revealing private information in a wide range of environments, from auctions to contests, public good provision, and quantity competition with private information. It is also a cornerstone of the theory: monotonicity is typically the condition that delivers well-behaved equilibria (existence and uniqueness), intuitive comparative statics, and tractable mechanism design. On the theoretical ground, Athey (2001) establishes seminal results on the existence of a monotone pure strategy Bayesian Nash equilibrium under the Spence-Mirrlees single-crossing condition, which are further extended to accommodate multi-dimensional type and action spaces by McAdams (2003), and to allow for more general spaces by Reny (2011). From an econometric perspective, monotone equilibrium strategies are equally central because they justify the inversion logic that maps observed actions back into latent types, enabling identification and inference in structural models. See, e.g., Guerre et al. (2000), Li et al. (2002) and a large related literature in first price auctions, Kline (2025) in allocation-transfer games, Li et al. (2025) and Li et al. (2026) for Bayesian games with continuous payoffs, and Aryal and Zincenko (2024) for Cournot competition with private information, to name only a few.

While there has been significant progress in identification and estimation of Bayesian games, there is remarkably little work on testing the monotonicity of Bayesian Nash equilibrium strategies, despite its role as a key equilibrium condition and as a cornerstone in identification and estimation. This gap is consequential because monotonicity is routinely imposed as an equilibrium restriction and as a prerequisite for inversion-based identification and estimation. If monotonicity fails in the data, the structural model and its implied

counterfactuals may be invalid. Moreover, monotonicity has independent empirical content in contexts where departures from competitive behavior are of interest. For example, Bajari and Ye (2003) derive a set of sufficient and necessary conditions for the symmetric and competitive bidding equilibrium including monotonicity of the bidding equilibrium, thus rejection of monotonicity can constitute a test for collusion. However, they bypass testing for monotonicity, and only test for other conditions such as exchangeability, largely because tools for testing monotone equilibrium strategies have been absent.

Testing monotone equilibrium strategies in types is challenging because types are unobserved to the econometrician. To date, the only development is in the auction setting considered in Liu and Vuong (2021), who exploit the equivalence between monotonicity of Bayesian Nash equilibrium bidding strategies in private values and monotonicity of the quasi-inverse of the bidding strategy, which is identified from observed bids.<sup>2</sup> They reformulate the test as one for the concavity of the integrated quasi-inverse strategy to avoid nonparametric estimation of bid densities in the quasi-inverse formulation. They then construct a statistic based on the difference between this function and its least concave majorant (LCM).

This paper aims to fill the gap in the literature by developing a new and easy-to-implement test for monotone equilibrium strategies in general Bayesian games. To this end, we first characterize the sufficient and necessary conditions for the existence of a private type distribution that can rationalize the observed action distribution in general Bayesian games considered in Athey (2001) within the symmetric independent private type paradigm. As such, we extend the rationalization logic in the empirical auction literature to games of incomplete information, which include not only auctions that have discontinuous payoffs, but also other games with continuous payoffs such as contests, public good provision, and Cournot competition with private information, among others.<sup>3</sup>

Our rationalization result delivers an equivalence that forms the basis of our test: within

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<sup>2</sup>In a somewhat related yet different vein, noting that the nonparametric estimator of the quasi-inverse bidding strategy in Guerre et al. (2000) does not impose the monotonicity condition, Henderson et al. (2012) and Ma et al. (2021) develop monotonicity constrained nonparametric estimators of the quasi-inverse strategies.

<sup>3</sup>The rationalization result in the empirical auction literature was first established in Guerre et al. (2000) for the symmetric independent private value setting, and has been extended to the symmetric affiliated private value setting in Li et al. (2002), and the asymmetric affiliated private value setting in Campo et al. (2003), as well as for risk-averse bidders in Guerre et al. (2009).

the maintained framework, monotonicity of the equilibrium strategy in private types holds if and only if the quasi-inverse equilibrium strategy is monotone in actions. The quasi-inverse is a ratio of the derivatives of two reduced-form objects—the expected allocation rule and the expected payment rule—that are functions of observed actions and can therefore be identified from the distribution of actions given the known institutional mapping from actions to allocations and transfers. As a result, the testing problem becomes to test whether this identified quasi-inverse map is monotone in the action.

Although this insight is in a similar spirit to Liu and Vuong (2021), our testing approach differs fundamentally. First, we move beyond first-price auctions and develop a single and unified framework that covers a broad class of Bayesian games with continuous actions, including contests, public good provision, and Cournot competition with private information, among others. Second, rather than relying on an LCM-based concavity test, we reformulate the monotonicity null into a countable collection of moment inequalities that involve only unconditional expectations and can be estimated by simple sample analogs, following the approach of Hsu et al. (2019), Hsu and Shen (2021) and Hsu et al. (2025), who reformulate the null hypothesis of monotonicity in regressions, conditional treatment effects under regression discontinuity designs, and mean potential outcomes in continuous treatment effect models, respectively, to moment inequalities. Third, this moment-inequality formulation naturally extends to heterogeneous games, allowing us to control for game-specific covariates and to develop procedures that allow for both observed and unobserved game heterogeneity—features that are essential in empirical work and difficult to accommodate with LCM-type approaches.

Building on this moment-inequality representation, we construct a Cramér–von Mises (CvM) type statistic based on the estimated inequalities. We combine a nonparametric bootstrap and the generalized moment selection method of Andrews and Shi (2013) to construct the critical values. We show that our test controls size asymptotically and is consistent against any fixed alternative. We also provide a practical guide by specializing the general procedure to four widely used models: first-price auctions, Tullock contests, public good provision, and Cournot competition with private information. Monte Carlo evidence indicates that the test performs well in finite samples, including sample sizes typical in auction applications.

We apply our test to the asphalt paving procurement datasets studied in Aaltio et al. (2025), where cartel detection is a central concern. The empirical findings are consistent with the evidence reported in Aaltio et al. (2025): we do not reject monotonicity in the dataset where no collusion was found, while in datasets where collusion was detected, our results provide evidence against monotonicity. To our knowledge, this is the first empirical application to formally test monotonicity of equilibrium strategies in a Bayesian Nash equilibrium framework using real data.

An important advantage of our moment inequalities-based test for monotonicity is it naturally accommodates heterogeneous games. Controlling for game-specific covariates is essential in almost all empirical applications. Moreover, our approach can accommodate unobserved game heterogeneity, noting that it has been an important and yet challenging issue in the structural auction literature to develop methods in identification and inference with unobserved auction heterogeneity.<sup>4</sup> Specifically, We develop fully nonparametric tests that can control for observed heterogeneity and also provide tests based on implications that remain valid under unobserved heterogeneity. Because fully nonparametric procedures can suffer from the curse of dimensionality, we additionally propose semiparametric tests based on action homogenization, extending the bid homogenization method proposed in Haile et al. (2003) beyond auctions to general Bayesian games. As such, our paper provides a monotonicity testing framework for incomplete-information games that can accommodate both observed and unobserved heterogeneity in a way suitable for applied work. .

Our work contributes to the econometrics of games of incomplete information. Our focus is on Bayesian games with continuous actions, which have witnessed vast development in auctions, and recent emerging literature in other games. See Larsen and Zhang (2018), Agarwal et al. (2023), Kline (2025), Li et al. (2025), and Li et al. (2026) for identification of general Bayesian games with different payoff/information structures; He and Huang (2021) for estimation of a contest model, Aryal and Gabrielli (2020) for estimation of a competitive nonlinear pricing model, and Aryal and Zincenko (2024) for estimation of a Cournot

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<sup>4</sup>While it is important to control for unobserved game heterogeneity, little attention has been paid to controlling for unobserved heterogeneity in Bayesian games other than auctions. For identification and estimation of auction models with unobserved heterogeneity, see Li and Zheng (2009), Krasnokutskaya (2011), Roberts (2013), Compiani et al. (2020), and Luo et al. (2024), among others, and Haile and Kitamura (2019) for a survey.

competition model with private information.<sup>5</sup>

Our work also contributes to the econometric literature on testing shape restrictions, and monotonicity in particular.<sup>6</sup> The recent projection-based approach of Fang and Seo (2021), for example, provides a general framework for testing shape restrictions defined by convex cones. As Liu and Vuong (2021) emphasize, most existing tests focus on monotonicity in observed variables, whereas both their work and ours address monotonicity in latent variables. While Liu and Vuong (2021) rely on the LCM in the spirit of Fang and Seo (2021), our test employs a transformation that leads to moment inequalities akin to Hsu et al. (2019), Hsu and Shen (2021) and Hsu et al. (2025); the advantages of our test highlight a flexible route to testing shape restrictions in structural models and demonstrate the practical value of such methods in applications.

This paper is organized as follows. Section 2 characterizes differentiable symmetric equilibrium strategies within the symmetric independent private-type paradigm and establishes the rationalization result linking monotonicity in types to monotonicity of the quasi-inverse map in actions. It then develops our benchmark test for homogeneous games and derives its asymptotic properties. Section 3 illustrates implementation of our benchmark test in auctions, contests, public good provision, and Cournot competition with private information. Section 4 considers the extensions of our tests in Section 2 to allow for heterogeneous games, controlling for observed and unobserved game heterogeneity. Section 5 reports Monte Carlo evidence. Section 6 presents the asphalt paving procurement auction application. Section 7 concludes. Technical proofs are collected in the Appendix.

## 2 Games of Incomplete Information

In this section, we first characterize the sufficient and necessary conditions for the existence of a private type distribution that can rationalize the observed action distribution in general Bayesian games considered in Athey (2001) within the symmetric independent private type

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<sup>5</sup>Notably, there has been significant development in the econometrics of games of incomplete information with discrete actions, e.g., Seim (2006), Aradillas-Lopze (2010), Tang (2010), de Paula and Tang (2012), Wan and Xu (2014), Aguirregabiria and Mira (2019), among others.

<sup>6</sup>See, e.g., Chetverikov et al. (2018) for a review on the econometrics of shape restrictions.

paradigm, in which all agents play a symmetric, monotone and differentiable strategy. The framework here covers a large class of Bayesian games, including the trading games with auction games as special cases in Larsen and Zhang (2018), the allocation-transfer games considered in Kline (2025), which include auctions, contests, and public good provision, and general Bayesian games with continuous payoffs in Li et al. (2025), which include contests, public good provision, and Cournot competition with incomplete information.

## 2.1 Strictly Monotone Bayesian Nash Equilibrium and Rationalization Result

We consider a model with symmetric independent private types. We impose the following conditions on the types.

**Assumption 2.1** *Assume that (i) Agent  $i$ ,  $i = 1, \dots, N$ , draws his/her type independently from the other agents from a CDF  $F(\cdot)$  with the PDF  $f(\cdot)$ . (ii)  $f(\cdot)$  is strictly positive and bounded away from zero on its support, a compact interval  $\mathcal{V} = [\underline{v}, \bar{v}] \subseteq R$ , and is twice continuously differentiable on  $(\underline{v}, \bar{v})$ .*

Each agent only observes his/her own type, but not the other agents' types, while the number of agents  $N$ , and  $F(\cdot)$  ( $f(\cdot)$ ) are common knowledge to all agents, who are assumed to be risk neutral. Let  $D_i$  be the allocation indicating  $i$  attaining the portion of the good or the probability of attaining the good, and let  $C_i$  be any net payment made or cost incurred by  $i$ . The outcome allocations and transfers for all agents are calculated as functions of all agents' actions and we introduce an individual allocation function  $D_i \equiv D(B_i, B_{-i})$ , which determines the allocation of the item for a given vector of actions  $B_1, \dots, B_N$ . The eventual payment  $C_i$  is also a function of all agents' actions and thus we introduce an individual payment function  $C(B_i, B_{-i})$  such that  $C_i \equiv C(B_i, B_{-i})$ , which is the final payment made by agent  $i$ . The outside option is normalized to zero. If agent  $i$  has type  $V_i$ , agent  $i$ 's utility is linear in type:  $V_i D_i - C_i$ .

Having observed their  $V_i$ 's, agents choose a monotone strategy mapping values into an action  $b \in \mathcal{B}$ , where  $\mathcal{B}$  is the space of actions available to agents. We assume that  $\mathcal{B} = [b, \bar{b}]$

with  $0 \leq \underline{b} < \bar{b} < \infty$ . For a given strategy  $s$ , we write  $s(v)$  for the action played by type  $v$  under  $s(\cdot)$ .

We define  $P(b)$  and  $T(b)$  to be the expected outcome allocations and expected payment, respectively, for agent  $i$  when agent  $i$  plays action  $b$  that is integrated over the distribution of rivals' strategies  $B_{-i}$ . Specifically,

$$P(b) = E_{B_{-i}}[D_i(B_i, B_{-i})|B_i = b], \quad T(b) = E_{B_{-i}}[C_i(B_i, B_{-i})|B_i = b]. \quad (2.1)$$

We make the following assumption on the smoothness of both  $P(\cdot)$  and  $T(\cdot)$ .

**Assumption 2.2** *Both  $T(b)$  and  $P(b)$  are twice continuously differentiable in  $b$  on  $(\underline{b}, \bar{b})$ .*

A similar assumption is made in Larsen and Zhang (2018) for point identification of trading games and Kline (2025) for point identification of allocation-transfer games, respectively.

Now define the interim expected payoff for an agent with type  $v$  and action  $b$  as  $\Pi(b, v) \equiv vP(b) - T(b)$ . we make the following assumptions:

**Assumption 2.3**  $\frac{\partial^2 \Pi(b, v)}{\partial b \partial v} > 0$  for all  $v \in (\underline{v}, \bar{v})$  and  $b \in (\underline{b}, \bar{b})$ .

**Assumption 2.4**  $\frac{\partial^2 \Pi(b, v)}{\partial b^2} < 0$  for all  $v \in (\underline{v}, \bar{v})$  and  $b \in (\underline{b}, \bar{b})$ .

Assumption 2.3 is the strict supermodularity assumption, which is stronger than the single crossing condition in Athey (2001) who establishes existence of monotone Bayesian Nash equilibrium in games of incomplete information with continuous action spaces. With the specification of  $\Pi(b, v)$  as  $vP(b) - T(b)$ , Assumption 2.3 is equivalent to  $P'(b) > 0$ , which means that the expected outcome allocation is strictly increasing in actions. Assumption 2.4 assumes the expected payoff to be strictly concave. These two assumptions are needed to get strictly monotone Bayesian Nash equilibrium (SMBNE) in our framework where we use the first order condition (FOC) approach in considering differentiable equilibrium strategies.

**Lemma 2.1** *Under Assumptions 2.1, 2.2, 2.3 and 2.4,  $s'(v) > 0$  for all  $v \in (\underline{v}, \bar{v})$ .*

For an SMBNE where the FOC is satisfied, each agent's strategy  $s(v_i)$  needs to be a best response given that all rival players play according to  $s(v)$  for all  $v \in [\underline{v}, \bar{v}]$ . This means that

incentive compatibility holds:

$$s(v_i) \in \arg \max_{b' \in \mathcal{B}} v_i P(b') - T(b'). \quad (2.2)$$

Suppose that the SMBNE is an interior solution. The FOC-based lemma below applies on intervals where best responses are in the interior and  $s(\cdot)$  is differentiable.

With agents behaving non-cooperatively, a common strategy  $s_i = s(v_i)$  is an equilibrium strategy if, when adopted by all agents but one, say agent 1, it is optimal for that agent to adopt the given monotone strategy as well. Hence, we can write any action as  $s_1 = s(x)$  and view agent 1 as choosing  $x$ . It then follows that  $s(v)$  is an equilibrium strategy if agent 1 can do no better than choosing  $x = v_1$ , and so its action is  $s(v_1)$ . To this end, the expected gain to agent 1 can be expressed as

$$\Pi(x, v_1) = v_1 P(s(x)) - T(s(x)).$$

Thus, the first-order condition for an interior optimum for any agent with valuation  $v$  is to choose  $x = v$ , such that

$$\frac{\partial \Pi}{\partial x}(x, v) = [v P'(s(x)) - T'(s(x))] s'(x) = 0 \quad \text{at } x = v.$$

Because  $s'(x) > 0$ , we get

$$v = \frac{T'(s(v))}{P'(s(v))} \equiv \xi(b), \quad (2.3)$$

where  $b = s(v)$ .

(2.3) is established in Corollary 2 in Larsen and Zhang (2018) for trading games, and in equation (26) in Kline (2025) for allocation-transfer games in the independent valuation paradigm. We refer to  $\xi(\cdot)$  in (2.3) as the quasi-inverse equilibrium strategy in all the Bayesian games we consider.

For a continuously differentiable CDF  $G$  on  $[\underline{b}, \bar{b}]$  with density  $g > 0$  on  $(\underline{b}, \bar{b})$ , let  $P(\cdot; G)$  and  $T(\cdot; G)$  denote the reduced-form expected allocation and transfer when opponents' ac-

tions are i.i.d. with the CDF  $G(\cdot)$ , and assume  $P(\cdot; G), T(\cdot; G) \in C^2(\underline{b}, \bar{b})$  with  $P'(b; G) > 0$  for all  $b \in (\underline{b}, \bar{b})$ . Define

$$\xi(b; G) := \frac{T'(b; G)}{P'(b; G)}, \quad b \in (\underline{b}, \bar{b}),$$

and assume  $\xi(\cdot; G)$  extends continuously to  $[\underline{b}, \bar{b}]$  with  $\underline{\xi} := \xi(\underline{b}; G)$  and  $\bar{\xi} := \xi(\bar{b}; G)$ .

**Theorem 2.1 (Rationalizability and Quasi-Inverse Equilibrium Strategy)** *Fix a continuously differentiable CDF  $G(\cdot)$  on  $[\underline{b}, \bar{b}]$  with density  $g > 0$  on  $(\underline{b}, \bar{b})$ . Then  $G$  is rationalizable by a strictly monotone, symmetric, and differentiable equilibrium if and only if the following (C1) and (C2) hold.*

(C1) *Actions  $(B_1, \dots, B_N)$  are i.i.d. with common CDF  $G(\cdot)$ .*

(C2)  *$\xi(\cdot; G)$  is strictly increasing on  $(\underline{b}, \bar{b})$ .*

*In addition, for any strictly monotone interior symmetric equilibrium rationalizing  $G(\cdot)$ , the type distribution  $F(\cdot)$  is unique.*

Theorem 2.1 extends Theorem 1 in Guerre et al. (2000) in first price auctions within the symmetric independent private value paradigm to general Bayesian games considered in Athey (2001) within the symmetric independent private type paradigm and with differentiable equilibrium strategies. First, it provides necessary and sufficient conditions for the existence of a private type distribution that can rationalize the observed action distribution. These two conditions are independence of actions and also monotonicity of the quasi-inverse equilibrium strategy, with the latter being the basis for our tests for monotone equilibrium strategies. Second, it establishes nonparametric point identification of the private type distribution from the distribution of the observed actions in the model considered. Our result extends the point identification result in Li et al. (2026) in Bayesian games with continuous payoff functions to general Bayesian games. Relatedly, under somewhat different set of assumptions, Kline (2025) establishes the point identification result in allocation-transfer games.

## 2.2 Proposed Test

Based on Theorem 2.1, we formulate the null hypothesis as

$$H_0 : \xi(b) = \frac{T'(b)}{P'(b)} \text{ is weakly increasing in } b. \quad (2.4)$$

In (2.4), we suppress the dependence of  $\xi$ ,  $T$  and  $P$  functions on  $G$  for notational simplicity. Ideally, the null should be that  $\xi(b)$  is strictly increasing, which usually results in a degenerate case for the asymptotic distribution of the test statistics. Therefore, we include the equality in the null hypothesis, as is usually done in the literature, e.g., Liu and Vuong (2021).<sup>7</sup> We propose a test for the null hypothesis in (2.4) and our test is similar to those in Hsu et al. (2019), Hsu and Shen (2021) and Hsu et al. (2025). We first adapt the transformation in Lemma 3.1 of Hsu and Shen (2021) to transform the null hypothesis in (2.4) to countably many inequalities without loss of information and we summarize this in the following lemma. Define  $a = \bar{b} - \underline{b}$ , which is the length of  $[\underline{b}, \bar{b}]$ .

**Lemma 2.2** *Let  $h(b)$  be a known function such that  $0 < h(b)P'(b) \leq M < \infty$ . Then  $H_0$  in (2.4) is equivalent to*

$$H'_0 : \nu(b_1, b_2, q) = M(b_2, q)W(b_1, q) - M(b_1, q)W(b_2, q) \leq 0 \text{ for all } (b_1, b_2, q) \in \mathcal{L}, \quad (2.5)$$

where

$$M(b, q) = \int_b^{b+\frac{a}{q}} h(\tilde{b})T'(\tilde{b})d\tilde{b}, \quad W(b, q) = \int_b^{b+\frac{a}{q}} h(\tilde{b})P'(\tilde{b})d\tilde{b}, \text{ and} \quad (2.6)$$

$$\mathcal{L} = \left\{ (b_1, b_2, q) : \left( \frac{b_1 - \underline{b}}{a}, \frac{b_2 - \underline{b}}{a} \right) \cdot q \in (0, 1, \dots, q)^2, b_1 > b_2 \text{ for } q=2,3,\dots \right\}. \quad (2.7)$$

Lemma 2.2 shows that the transformation of the null hypothesis  $H_0$  to  $H'_0$  in (2.5) which includes a countable many moment inequalities is without loss of information. The  $h(b)$  function can be any function that satisfies  $0 < h(b)P'(b) \leq M < \infty$ , therefore, in practice,

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<sup>7</sup>Another reason to consider weakly increasing in the null hypothesis is that any test of the null of strictly increasing is likely to have power equal to size against alternatives that are weakly (but not strictly) increasing. This is related to the problems in Romano (2004). We thank Andres Santos for pointing this out.

$h(b)$  can be case specific and one can pick  $h(b)$  in a way such that  $\nu(b_1, b_2, q)$  can be estimated and weakly converges to a Gaussian process.

Specifically, let  $\widehat{M}(b, q)$  and  $\widehat{W}(b, q)$  denote the estimators for  $M(b, q)$  and  $W(b, q)$ , respectively. Denote the estimator for  $\nu(b_1, b_2, q)$  as

$$\widehat{\nu}(b_1, b_2, q) = \widehat{M}(b_2, q)\widehat{W}(b_1, q) - \widehat{M}(b_1, q)\widehat{W}(b_2, q). \quad (2.8)$$

**Assumption 2.5** *Assume that we observe actions  $\{B_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$  that are i.i.d. random variables across  $L$  games and the agents in each game.*

Assumption 2.5 imposes conditions on the data that econometricians observe. Let  $S = NL$  denote the total number of observed actions.

**Assumption 2.6** *Assume that*

$$\sqrt{S}(\widehat{\nu}(\cdot, \cdot, \cdot) - \nu(\cdot, \cdot, \cdot)) \Rightarrow \Phi_\kappa(\cdot, \cdot, \cdot)$$

in which  $\Rightarrow$  denotes weak convergence and  $\Phi_\kappa(\cdot, \cdot, \cdot)$  is a Gaussian process with covariance kernel being  $\kappa((b'_1, b'_2, q'), (b''_1, b''_2, q''))$  for  $(b'_1, b'_2, q'), (b''_1, b''_2, q'') \in \mathcal{L}$ . In addition,  $\kappa$  is not a zero function.

Assumption 2.6 requires that there exists a consistent estimator for  $\nu(\cdot)$  which weakly converges to a Gaussian process with proper rescaling. In addition, that  $\kappa$  is not a zero function implies that  $\widehat{\nu}(\cdot)$  is not degenerate. Define  $\sigma_\nu^2(b_1, b_2, q) = \kappa((b_1, b_2, q), (b_1, b_2, q))$  which is the asymptotic variance of  $\sqrt{S}(\widehat{\nu}(b_1, b_2, q) - \nu(b_1, b_2, q))$  and its estimator as  $\widehat{\sigma}_\nu^2(b_1, b_2, q)$

**Assumption 2.7** *Assume that  $\widehat{\sigma}_\nu^2(b_1, b_2, q)$  is uniformly consistent for  $\sigma_\nu^2(b_1, b_2, q)$  in that  $\sup_{(b_1, b_2, q) \in \mathcal{L}} |\widehat{\sigma}_\nu^2(b_1, b_2, q) - \sigma_\nu^2(b_1, b_2, q)| \xrightarrow{P} 0$ . In addition,  $\sigma_\nu^2(\underline{b}, (\underline{b} + \bar{b})/2, 2) > 0$ .*

Assumption 2.7 requires that there exists a uniformly consistent estimator for the asymptotic variance. We specifically assume that one of the moments is not degenerate. In practice, if the influence functions of the  $\widehat{\nu}(\cdot)$  are known, then one can use the estimated influence functions to construct an estimator for the variance function. Or one might use the bootstrap estimator to estimate the variance function.

For some small  $\epsilon > 0$ , define

$$\widehat{\sigma}_{\nu,\epsilon}^2(b_1, b_2, q) = \max\{\widehat{\sigma}_\nu^2(b_1, b_2, q), \epsilon \cdot \widehat{\sigma}_\nu^2(\underline{b}, (\underline{b} + \bar{b})/2, 2)\}. \quad (2.9)$$

For a weighted function  $Q(b_1, b_2, q) > 0$  and  $\sum_{(b_1, b_2, q) \in \mathcal{L}} Q(b_1, b_2, q) < \infty$ , define our test statistic as

$$\widehat{T}_S = \sum_{(b_1, b_2, q) \in \mathcal{L}} \max\left\{\sqrt{S} \frac{\widehat{\nu}(b_1, b_2, q)}{\widehat{\sigma}_{\nu,\epsilon}(b_1, b_2, q)}, 0\right\}^2 Q(b_1, b_2, q). \quad (2.10)$$

We propose a nonparametric bootstrap combining with the generalized moment selection (GMS) method introduced by Andrews and Soares (2010) and Andrews and Shi (2013) to construct the critical values for our test.<sup>8</sup> For the nonparametric bootstrap, we bootstrap games. That is, let  $\{(B_{1\ell}^*, \dots, B_{N\ell}^*) : \ell \leq L\}$  be a bootstrap sample in which  $B_{i,\ell}^* = B_{i\ell^*}$  and  $\{\ell^* : \ell \leq L\}$  is an i.i.d. bootstrap sample drawn from the empirical distribution of  $\{\ell : \ell \leq L\}$ . Define the bootstrap estimator for  $\nu(b_1, b_2, q)$  as

$$\widehat{\nu}^*(b_1, b_2, q) = \widehat{M}^*(b_2, q) \widehat{W}^*(b_1, q) - \widehat{M}^*(b_1, q) \widehat{W}^*(b_2, q), \quad (2.11)$$

where  $\widehat{M}^*(b, q)$  and  $\widehat{W}^*(b, q)$  are bootstrap estimators for  $M(b, q)$  and  $W(b, q)$ , respectively, based on the bootstrap sample. The bootstrapped process is defined as  $\Phi^*(\cdot, \cdot, \cdot) = \sqrt{S}(\widehat{\nu}^*(\cdot, \cdot, \cdot) - \widehat{\nu}(\cdot, \cdot, \cdot))$ .

**Assumption 2.8** *Assume that  $\Phi^*(\cdot, \cdot, \cdot) \Rightarrow \Phi_\kappa(\cdot, \cdot, \cdot)$  conditional on the sample path with probability approaching 1 where  $\Phi_\kappa(\cdot, \cdot, \cdot)$  is as in Assumption 2.6 and we denote it as  $\Phi^*(\cdot, \cdot, \cdot) \xrightarrow{P} \Phi_\kappa(\cdot, \cdot, \cdot)$ .*

Assumption 2.8 requires that we have a bootstrap method that approximates the limiting process  $\Phi_\kappa(\cdot, \cdot, \cdot)$  well.

As most papers in the moment inequality literature, we use the GMS method to construct the critical value. By doing this, one can construct a more powerful test without resorting to

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<sup>8</sup>The GMS method is similar to the recentering method in Hansen (2005) and Donald and Hsu (2016), as well as the contact set approach in Linton et al. (2010).

the least favorable configuration. Let  $\{\beta_\ell, \ell = 1, 2, \dots\}$  and  $\{\kappa_\ell, \ell = 1, 2, \dots\}$  be sequences of positive numbers. Define the GMS function  $\psi_S(b_1, b_2, q)$  as

$$\psi_S(b_1, b_2, q) = -\beta_S \cdot 1\left(\frac{\sqrt{S}\widehat{\nu}(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} < -\kappa_S\right). \quad (2.12)$$

We impose conditions on  $\beta_S$  and  $\kappa_S$ .

**Assumption 2.9** *Assume that (i)  $\lim_{S \rightarrow \infty} \kappa_S = \infty$  and  $\lim_{S \rightarrow \infty} \kappa_S/\sqrt{S} = 0$  and (ii)  $\beta_S$  is non-decreasing,  $\lim_{S \rightarrow \infty} \beta_S = \infty$  and  $\lim_{S \rightarrow \infty} \beta_S/\kappa_S = 0$ .*

For a significance level  $\alpha < 1/2$ , define the bootstrapped critical value  $\hat{c}_\eta$  as

$$\hat{c}_\eta = \sup \left\{ c \mid P^* \left( \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi^*(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} + \psi_S(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q) \leq c \right) \leq 1 - \alpha + \eta \right\} + \eta,$$

where  $\eta > 0$  is an arbitrarily small positive number. Note that  $\hat{c}_\eta$  is the  $(1 - \alpha + \eta)$ -th quantile of the simulated null distribution plus  $\eta$  which is called an infinitesimal uniformity factor in Andrews and Shi (2013). Following Hsu et al. (2019), we set  $\beta_S = 0.85 \cdot \ln(S)/\ln \ln(S)$ ,  $\kappa_S = 0.15 \cdot \ln(S)$  and  $\eta = 10^{-6}$ .

The decision rule is: “Reject  $H_0$  ( $H'_0$ ) when  $\widehat{T}_S > \hat{c}_\eta$ .”

The following theorem shows the asymptotic size control and the power against fixed alternatives of our test.

**Theorem 2.2** *Suppose Assumptions 2.5, 2.6, 2.7, 2.8, 2.9 hold. Then,*

- (a) *under  $H_0$ ,  $\lim_{S \rightarrow \infty} P(\widehat{T}_S > \hat{c}_\eta) \leq \alpha$ ;*
- (b) *under  $H_1$ ,  $\lim_{S \rightarrow \infty} P(\widehat{T}_S > \hat{c}_\eta) = 1$ .*

Theorem 2.2 shows that our test has asymptotic size control under the null hypothesis and is consistent against fixed alternatives.

**Remark 2.1** *In practice, the data may include games with varying numbers of agents. In the Appendix, we discuss how to conduct tests that allow for such heterogeneity. The key issue is that the quasi-inverse equilibrium strategy, the  $\xi$  function, implicitly depends on the number of agents in a game. If this strategy varies with the number of agents, then the*

corresponding moment conditions must be treated separately for each group. Accordingly, when games involve different numbers of agents, we implement a joint test across these groups. For further details, see Appendix B.

### 3 Examples

In this section, we consider four examples of Bayesian games. The first example is the first-price sealed-bid auction model within the symmetric independent private value paradigm, the second one is a Tullock contest model, the third one is a public good provision model, and the last one is the Cournot competition model with incomplete information.

#### 3.1 First-price Sealed-bid Auction Model

As the first example, we consider a first-price sealed-bid auction model for homogeneous goods with a fixed number of bidders. The bids submitted by the bidders are the observed actions in each auction. The econometricians observe  $N$  bids  $(B_{1\ell}, \dots, B_{N\ell})$  in the  $\ell$ -th auction for  $\ell = 1, \dots, L$ , but not bidders' valuations be  $\{V_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$ . Thus, the allocation function of the auctioned product for player  $i$  is determined by the indicator function  $D(B_i, B_{-i})$  defined as

$$D(B_i, B_{-i}) = \begin{cases} 1, & \text{if } B_i > B_j \quad \forall j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

Also, the payment function of the auctioned product for player  $i$  is determined by the cost function defined as  $C(B_i, B_{-i}) = B_i \cdot D(B_i, B_{-i})$ . Integrating over the distribution of  $B_{-i}$  gives the monotone equilibrium expected allocation for bidder  $i$ , i.e.,  $E_{B_{-i}}[D(B_i, B_{-i})]$ . Let  $V_i$  be the value for agent  $i$ . Then for agent  $i$ , his/her expected equilibrium payoff  $\pi(B_i, V_i)$  is

$$\pi(B_i, V_i) = V_i \cdot E_{B_{-i}}[D(B_i, B_{-i})] - B_i \cdot E_{B_{-i}}[D(B_i, B_{-i})] = V_i P(B_i) - T(B_i).$$

The following assumption is summarized from Guerre et al. (2000) on the distribution of the valuations, which gives rise to the symmetric independent private value paradigm.

**Assumption 3.1** Assume that (i) the unobserved valuations  $\{V_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$  are i.i.d. with probability density function (PDF),  $f(\cdot)$ , and cumulative distribution function (CDF),  $F(\cdot)$ ; (ii)  $f(\cdot)$  is strictly positive and bounded away from zero on a convex and compact support,  $[\underline{v}, \bar{v}] \subseteq R$ , and is twice continuously differentiable on  $(\underline{v}, \bar{v})$ .

Assume that the observed bids are generated from the valuations satisfying Assumption 3.1 and by a BNE bidding strategy,  $s(V_{i,\ell})$ . It then follows that

$$B_{i,\ell} = s(V_{i,\ell}) \equiv V_{i,\ell} - \frac{1}{F(V_{i,\ell})^{N-1}} \int_{\underline{v}}^{V_{i,\ell}} F(u)^{N-1} du, \quad (3.1)$$

and the observed bids  $\{B_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$  are i.i.d. with PDF  $g(\cdot)$  and CDF  $G(\cdot)$  with a convex and compact support,  $[\underline{b}, \bar{b}] \subseteq R$ .<sup>9</sup> The BNE requires that the bidding strategy  $s(\cdot)$  is strictly increasing.<sup>10</sup>

It can be shown that  $P(b) = G(b)^{N-1}$  and  $T(b) = bG(b)^{N-1}$ . As a result,  $\xi(\cdot)$  in (2.3) becomes the quasi-inverse bidding strategy in Guerre et al. (2000):

$$\xi(b) = b + \frac{1}{N-1} \frac{G(b)}{g(b)}, \quad (3.2)$$

where  $G(\cdot)$  and  $g(\cdot)$  are the CDF and PDF of bids, respectively. Guerre et al. (2000) show that the observed bid distribution  $G(\cdot)$  is an equilibrium bid distribution if and only if the quasi-inverse strategy,  $\xi(b)$ , in (3.2) is strictly increasing, which is implied by Theorem 2.1.

Therefore, we define the null hypothesis as

$$H_{0,au} : \xi(b) = b + \frac{1}{N-1} \frac{G(b)}{g(b)} \text{ is weakly increasing in } b. \quad (3.3)$$

We apply Lemma 2.2 to transform the null hypothesis in (3.3) by setting  $h(b) = G(b)^{2-N}(N-1)^{-1}$ , so we have

$$M(b, q) = \int_b^{b+\frac{q}{N-1}} h(\tilde{b}) T'(\tilde{b}) d\tilde{b} = \int_b^{b+\frac{q}{N-1}} (\tilde{b}g(\tilde{b}) + (N-1)^{-1}G(\tilde{b})) d\tilde{b},$$

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<sup>9</sup>Note that  $g(\cdot)$  and  $G(\cdot)$  depend on  $N$ , the number of bidders in the auction, because the equilibrium bidding strategy in (3.1) is a function of  $N$ .

<sup>10</sup>Moreover, Guerre et al. (2000) show that  $s(\cdot)$  is at least three times continuously differentiable on  $(\underline{v}, \bar{v})$ .

$$W(b, q) = \int_b^{b+\frac{a}{q}} h(\tilde{b})P'(\tilde{b})d\tilde{b} = \int_b^{b+\frac{a}{q}} g(\tilde{b})d\tilde{b}.$$

Note that  $M(b, q)$  and  $W(b, q)$  are both identified and can be expressed as unconditional means. We summarize the results in the following lemma.

**Lemma 3.1** *The  $H_{0,au}$  in (3.3) is equivalent to*

$$H'_{0,au} : \nu(b_1, b_2, q) = M(b_2, q)W(b_1, q) - M(b_1, q)W(b_2, q) \leq 0 \text{ for all } (b_1, b_2, q) \in \mathcal{L}, \quad (3.4)$$

where

$$M(b, q) = E\left[B_{i,\ell}1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right] \\ + \frac{1}{N-1}E\left[1\left(B_{i,\ell} \leq b + \frac{a}{q}\right)\left(b + \frac{a}{q} - B_{i,\ell}\right) - 1(B_{i,\ell} \leq b)(b - B_{i,\ell})\right], \quad (3.5)$$

$$W(b, q) = E\left[1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right]. \quad (3.6)$$

Based on (3.5) and (3.6), we can apply the test in Section 2.2. In Appendix, we provide a uniform consistent estimator for the asymptotic variance for each moment and show the sufficient conditions hold for Theorem 2.2 in our case.

**Remark 3.2 (Procurement Auctions)** *The example above has considered high-bid first-price auctions to be in line with the literature where the identification results starting Guerre et al. (2000) have been established for high-bid auctions. For low-bid first-price auctions such as procurement auctions considered in our empirical application, we can similarly show that  $P(b) = (1 - G(b))^{N-1}$  and  $T(b) = b(1 - G(b))^{N-1}$ . Therefore, it is straightforward to see that*

$$\xi(b) = b - \frac{1}{N-1} \frac{1 - G(b)}{g(b)}.$$

In this case, the  $M(b, q)$  and  $W(b, q)$  are expressed as

$$M(b, q) = E\left[B_{i,\ell}1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right] - \frac{1}{N-1} \frac{a}{q} \\ + \frac{1}{N-1}E\left[1\left(B_{i,\ell} \leq b + \frac{a}{q}\right)\left(b + \frac{a}{q} - B_{i,\ell}\right) - 1(B_{i,\ell} \leq b)(b - B_{i,\ell})\right], \quad (3.7)$$

$$W(b, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \right], \quad (3.8)$$

respectively.

### 3.2 Tullock Contest Model

In the context of a Tullock contest such as in Tullock (1980), Ryvkin (2010) and Ewerhart (2014), let the contestants' valuations be  $\{V_{i,\ell} : i = 1, \dots, N\}$  which are unobservable to the econometrician. The efforts exerted by contestants are the observed actions. To be specific,  $N$  contestants choose effort levels  $B_i \geq 0$  to compete for a prize. The winning probability, conditional on rivals' effort levels  $B_{-i}$ , is given by the symmetric contest–success function

$$D(B_i, B_{-i}) = \frac{B_i}{\sum_{s=1}^N B_s}. \quad (3.9)$$

Also, the contestant  $i$  has a linear effort cost  $C(B_i, B_{-i}) = B_i$ . Integrating over the distribution of  $B_{-i}$  gives the monotone equilibrium expected allocation  $P(B_i) = E_{B_{-i}}[D(B_i, B_{-i})]$  and expected effort cost  $T(B_i) = E_{B_{-i}}[C(B_i, B_{-i})] = B_i$ . The expected payoff is

$$\pi(B_i, V_i) = V_i E_{B_{-i}}[D(B_i, B_{-i})] - E_{B_{-i}}[C(B_i, B_{-i})] = V_i P(B_i) - T(B_i).$$

Let the contestants' valuations be  $\{V_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$  which are unobservable to the econometrician. We impose the following assumption.

**Assumption 3.2** *Assume that (i) the unobserved valuations  $\{V_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$  are i.i.d. with probability density function (PDF),  $f(\cdot)$ , and cumulative distribution function (CDF),  $F(\cdot)$ ; (ii)  $f(\cdot)$  is strictly positive and bounded away from zero on a convex and compact support,  $[\underline{v}, \bar{v}] \subseteq R$ , and is twice continuously differentiable on  $(\underline{v}, \bar{v})$ .*

Ewerhart and Quartieri (2020) show the existence and uniqueness of a monotone equilibrium. The FOC for an interior optimum is

$$\frac{\partial \Pi(B_i; V_i)}{\partial B_i} = V_{i,\ell} P'(B_i) - 1 = 0 \quad \Rightarrow \quad V_i = \frac{1}{P'(B_i)}.$$

In this case, we have  $P(b) = E_{B_{-i}}[D(b, B_{-i})]$  and  $T(b) = b$ . It is straightforward to see that  $P'(b) = b^{-1}E_{B_{-i}}[D(b, B_{-i})(1 - D(b, B_{-i}))]$  which is strictly positive. We then have

$$\xi(b) = \frac{T'(b)}{P'(b)} = \frac{1}{b^{-1}E_{B_{-i}, \ell}[D(b, B_{-i})(1 - D(b, B_{-i}))]}$$

which is the same as in Ewerhart and Quartieri (2020). Therefore, we define the null hypothesis as

$$H_{0, con} : \xi(b) = \frac{1}{b^{-1}E_{B_{-i}}[D(b, B_{-i})(1 - D(b, B_{-i}))]} \text{ is weakly increasing in } b. \quad (3.10)$$

We apply Lemma 2.2 to transform the null hypothesis in (3.10) by setting  $h(b) = bg(b)$  with  $g(b)$  being the density function of the actions under BNE, so we have

$$\begin{aligned} M(b, q) &= \int_b^{b+\frac{a}{q}} h(\tilde{b})T'(\tilde{b})d\tilde{b} = \int_b^{b+\frac{a}{q}} \tilde{b}g(\tilde{b})d\tilde{b}, \\ W(b, q) &= \int_b^{b+\frac{a}{q}} h(\tilde{b})P'(\tilde{b})d\tilde{b} = \int_b^{b+\frac{a}{q}} E_{B_{-i}}[D(\tilde{b}, B_{-i})(1 - D(\tilde{b}, B_{-i}))]g(\tilde{b})d\tilde{b}. \end{aligned}$$

Note that  $M(b, q)$  and  $W(b, q)$  are both identified and can be expressed as unconditional means. We summarize the results in the following lemma.

**Lemma 3.2** *The  $H_{0, con}$  in (3.10) is equivalent to*

$$H'_{0, con} : \nu(b_1, b_2, q) = M(b_2, q)W(b_1, q) - M(b_1, q)W(b_2, q) \leq 0 \text{ for all } (b_1, b_2, q) \in \mathcal{L}, \quad (3.11)$$

where

$$M(b, q) = E\left[B_{i, \ell} \cdot 1\left(b \leq B_{i, \ell} \leq b + \frac{a}{q}\right)\right] \quad (3.12)$$

$$W(b, q) = E\left[1\left(b \leq B_{i, \ell} \leq b + \frac{a}{q}\right) \frac{B_{i, \ell}}{\sum_{j=1}^N B_{j, \ell}} \left(1 - \frac{B_{i, \ell}}{\sum_{j=1}^N B_{j, \ell}}\right)\right]. \quad (3.13)$$

Based on (3.12) and (3.13), we are able to apply the test in Section 2.2.

### 3.3 Public Good Provision Model

Suppose there are  $N$  contributors, and contributor  $i$  voluntarily contributes  $B_i$  to provide a public good. Let  $\Omega$  be the common benefit function of the total contributions. We assume that  $\Omega$  is known, and is strictly increasing and concave. Therefore, we assume  $D(B_i, B_{-i}) = \Omega(\sum_{s=1}^N B_s)$ . Also, if contributor  $i$  decides to contribute  $B_i$ , the net payment will be  $B_i$  for contributor  $i$ , so we have  $C(B_i, B_{-i}) = B_i$ . Contributor  $i$ 's private value of the public good  $V_i$  affects the marginal utility of consuming the public good. Then given the actions  $(B_1, \dots, B_N)$ , the utility for contributor  $i$  is  $V_i D(B_i, B_{-i}) - C(B_i, B_{-i})$ . Define  $P(b) = E_{B_{-i}}[D(b, B_{-i})]$  and  $T(b) = E_{B_{-i}}[C(b, B_{-i})] = B_i$ . Following Bag and Roy (2011) who extend Varian (1994)'s model with complete information to allow for incomplete information, the expected payoff for contributor  $i$  is

$$\pi(B_i, V_i) = V_i E_{B_{-i}}[D(B_i, B_{-i})] - B_i = V_i P(B_i) - T(B_i),$$

and we have

$$\xi(b) = \frac{T'(b)}{P'(b)} = \frac{1}{E_{B_{-i}}[\Omega'(b + \sum_{s \neq i} B_s)]}.$$

Therefore, we define the null hypothesis as

$$H_{0,pq} : \xi(b) = \frac{1}{E_{B_{-i}}[\Omega'(b + \sum_{s \neq i} B_s)]} \text{ is weakly increasing in } b. \quad (3.14)$$

We apply Lemma 2.2 to transform the null hypothesis in (3.14) by setting  $h(b) = g(b)$  with  $g(b)$  being the density function of the actions under BNE, so we have

$$\begin{aligned} M(b, q) &= \int_b^{b+\frac{a}{q}} h(\tilde{b}) T'(\tilde{b}) d\tilde{b} = \int_b^{b+\frac{a}{q}} g(\tilde{b}) d\tilde{b}, \\ W(b, q) &= \int_b^{b+\frac{a}{q}} h(\tilde{b}) P'(\tilde{b}) d\tilde{b} = \int_b^{b+\frac{a}{q}} E_{B_{-i}}[\Omega'(\tilde{b} + \sum_{s \neq i} B_s)] g(\tilde{b}) d\tilde{b}. \end{aligned}$$

Note that  $M(b, q)$  and  $W(b, q)$  are both identified and can be expressed as unconditional means. We summarize the results in the following lemma.

**Lemma 3.3** *The  $H_{0,pg}$  in (3.14) is equivalent to*

$$H'_{0,pg} : \nu(b_1, b_2, q) = M(b_2, q)W(b_1, q) - M(b_1, q)W(b_2, q) \leq 0 \text{ for all } (b_1, b_2, q) \in \mathcal{L}, \quad (3.15)$$

where

$$M(b, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \right] \quad (3.16)$$

$$W(b, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \Omega' \left( \sum_{j=1}^N B_{j,\ell} \right) \right]. \quad (3.17)$$

### 3.4 Cournot Competition Model with Private Information

For the last example, we consider testing in Cournot competition models. In the spirit of Vives (2002), let each firm  $i$  choose output  $B_i$  and have a private cost–efficiency parameter  $V_i$  and  $1/V_i$  denotes the cost for each unit produced for agent  $i$ . Let  $\mathcal{I}(\cdot)$  be the inverse demand function which is known.<sup>11,12</sup> Also we assume that  $\mathcal{I}(\cdot)$  is strictly decreasing and convex, so that the revenue  $D(B_i, B_{-i}) \equiv \mathcal{I}(B_i, B_{-i})B_i$  is strictly concave conditional on  $B_{-i}$ . Let  $C(B_i, B_{-i}) = B_i$  denote the units produced by agent  $i$ , so  $B_i/V_i$  is the total cost for agent  $i$ . Therefore, the total profit for agent  $i$  is  $D(B_i, B_{-i}) - C(B_i, B_{-i})/V_i$ . Integrating over the distribution of  $B_{-i}$ , the expected payoff can be rewritten as

$$\bar{\pi}(B_i, V_i) = E_{B_{-i}}[D(B_i, B_{-i})] - \frac{B_i}{V_i} \equiv P(B_i) - \frac{T(B_i)}{V_i},$$

where  $P(b) = E_{B_{-i}}[D(b, B_{-i})]$  and  $T(b) = b$ . Assume that the demand curvature is known to the researcher and small enough, e.g.,  $-2E[\mathcal{I}'] > E[\mathcal{I}'']b$  for all relevant  $b$ , where  $\mathcal{I}'$  and  $\mathcal{I}''$  denote the first and second derivatives, respectively. Note that since maximizing  $\bar{\pi}(b, V_i)$  over  $b$  is equivalent to maximizing  $\pi(b, V_i) \equiv V_i \cdot \bar{\pi}(b, V_i) = V_i P(B_i) - T(B_i)$  over  $b$ , we can

<sup>11</sup>Aryal and Zincenko (2024) extend Vives (2002) to allow for stochastic demand and a common technology shock.

<sup>12</sup>In some cases, the inverse demand function is unknown, but can be identified and estimated consistently from a separate sample. In this case, if the number of observations in the separate sample is large compared to the sample size of the testing sample so that we can ignore the estimation effect of the demand function asymptotically, then we can treat the estimated inverse demand function as a known function when we conduct the test.

still have the same form as before. Therefore, we have

$$\xi(b) = \frac{T'(b)}{P'(b)} = \frac{1}{E_{B_{-i}}[\mathcal{I}(b + \sum_{s \neq i}^N B_s) + \mathcal{I}'(b + \sum_{s \neq i}^N B_s)b]}.$$

Thus, we define the null hypothesis as

$$H_{0,cc} : \xi(b) = \frac{1}{E_{B_{-i}}[\mathcal{I}(b + \sum_{s \neq i}^N B_s) + \mathcal{I}'(b + \sum_{s \neq i}^N B_s)b]} \text{ is weakly increasing in } b. \quad (3.18)$$

We apply Lemma 2.2 to transform the null hypothesis in (3.18) by setting  $h(b) = g(b)$  with  $g(b)$  being the density function of the actions under BNE, so we have

$$\begin{aligned} M(b, q) &= \int_b^{b+\frac{a}{q}} h(\tilde{b})T'(\tilde{b})d\tilde{b} = \int_b^{b+\frac{a}{q}} g(\tilde{b})d\tilde{b}, \\ W(b, q) &= \int_b^{b+\frac{a}{q}} h(\tilde{b})P'(\tilde{b})d\tilde{b} = \int_b^{b+\frac{a}{q}} E_{B_{-i}}[\mathcal{I}(\tilde{b} + \sum_{s \neq i} B_s) + \mathcal{I}'(\tilde{b} + \sum_{s \neq i} B_s)b]g(\tilde{b})d\tilde{b}. \end{aligned}$$

Note that  $M(b, q)$  and  $W(b, q)$  are both identified and can be expressed as unconditional means. We summarize the results in the following lemma.

**Lemma 3.4** *The  $H_{0,cc}$  in (3.18) is equivalent to*

$$H'_{0,cc} : \nu(b_1, b_2, q) = M(b_2, q)W(b_1, q) - M(b_1, q)W(b_2, q) \leq 0 \text{ for all } (b_1, b_2, q) \in \mathcal{L}, \quad (3.19)$$

where

$$M(b, q) = E\left[1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right] \quad (3.20)$$

$$W(b, q) = E\left[1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\left(\mathcal{I}\left(\sum_{j=1}^N B_{j,\ell}\right) + \mathcal{I}'\left(\sum_{j=1}^N B_{j,\ell}\right)B_{i,\ell}\right)\right]. \quad (3.21)$$

## 4 Heterogeneous Games of Incomplete Information

In this section, we extend the test under homogeneous games of incomplete information in Section 2 to allow for heterogeneous games of incomplete information including observed

game heterogeneity and/or unobserved game heterogeneity. For both cases, we consider both nonparametric testing and semiparametric testing via homogenization.

## 4.1 Observed Game Heterogeneity: Nonparametric Test

We extend our test in Section 2 to control for the observed heterogeneity in the model. Specifically, let  $X$  with support  $\mathcal{X}$  denote a vector of covariates that is observed for each game. For example,  $X$  represents the characteristics of a game, such as the observed quality of a good. Without loss of generality, we assume that  $X$  is a scalar and  $\mathcal{X} = [0, 1]$ .<sup>13</sup> We impose the following conditions on the distribution of types.

**Assumption 4.1** *Assume that (i) Conditioning on  $X = x$ , agent  $i$ ,  $i = 1, \dots, N$ , draws his/her type independently from the other agents from a conditional CDF  $F(\cdot|x)$  with the conditional PDF  $f(\cdot|x)$ . (ii)  $f(\cdot|x)$  is strictly positive and bounded away from zero on its support, a compact interval  $\mathcal{V}_x = [\underline{v}_x, \bar{v}_x] \subseteq R$ , and is twice continuously differentiable on  $(\underline{v}_x, \bar{v}_x)$ .*

We define  $P(b, x)$  and  $T(b, x)$  to be the conditionally expected outcome allocations and conditionally expected payment, respectively, for agent  $i$  when agent  $i$  plays action  $b$  that is integrated over the distribution of rivals' strategies  $B_{-i}$  given  $X = x$ . Specifically,

$$\begin{aligned} P(b, x) &= E_{B_{-i}|X=x}[D_i(B_i, B_{-i})|B_i = b, X = x], \\ T(b, x) &= E_{B_{-i}|X=x}[C_i(B_i, B_{-i})|B_i = b, X = x]. \end{aligned} \quad (4.1)$$

Suppose all the conditions in Section 2.1 holds after we condition  $X = x$  for all  $x \in \mathcal{X}$ , then it is true that we can formulate the null hypothesis as

$$H_{0,x} : \xi(b, x) = \frac{T'(b, x)}{P'(b, x)} \text{ is weakly increasing in } b \text{ for all } x \in \mathcal{X}. \quad (4.2)$$

Adapting Lemma A.1 of Hsu and Shen (2021), we have the following equivalence result.

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<sup>13</sup>It is straightforward to allow for  $X$  being vectors of observed characteristics.

**Lemma 4.1** *Let  $h(b, x)$  be a known function such that  $0 < h(b, x)P'(b, x) \leq M < \infty$ . Then  $H_0$  in (4.2) is equivalent to*

$$H'_{0,x} : \nu(b_1, b_2, x, q) = M(b_2, x, q)W(b_1, x, q) - M(b_1, x, q)W(b_2, x, q) \leq 0$$

for all  $(b_1, b_2, x, q) \in \mathcal{L}_x$ ,

(4.3)

where

$$M(b, x, q) = \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{a}{q}} h(\tilde{b}, \tilde{x})T'(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x}$$

$$W(b, x, q) = \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{a}{q}} h(\tilde{b}, \tilde{x})P'(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x}, \text{ and}$$
(4.4)

$$\mathcal{L}_x = \left\{ (b_1, b_2, x, q) : \left( \frac{b_1 - b}{a}, \frac{b_2 - b}{a}, x \right) q \in (0, 1, \dots, q)^3, b_1 > b_2 \text{ for } q=2,3,\dots \right\}.$$
(4.5)

Once we get Lemma 4.1, the general test will be similar to that in Section 2.2. Therefore, we omit the details.

One interesting thing to note is that (4.2) actually implies the following:

$$\nu(b_1, b_2, q) = M(b_2, q)W(b_1, q) - M(b_1, q)W(b_2, q) \leq 0 \text{ for all } (b_1, b_2, q) \in \mathcal{L},$$

where

$$M(b, q) = \int_0^1 \int_b^{b+\frac{a}{q}} h(\tilde{b}, \tilde{x})T'(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x}, \quad W(b, q) = \int_0^1 \int_b^{b+\frac{a}{q}} h(\tilde{b}, \tilde{x})P'(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x}, \text{ and}$$

$$\mathcal{L} = \left\{ (b_1, b_2, q) : \left( \frac{b_1 - b}{a}, \frac{b_2 - b}{a} \right) \cdot q \in (0, 1, \dots, q)^2, b_1 > b_2 \text{ for } q=2,3,\dots \right\}.$$

This is equivalent to the null hypothesis  $H'_0$  in Lemma 2.2 and this is true by integrating out the distribution of  $X$  in Lemma 4.1 instead of restricting  $X$  in an interval. This will be the foundation for our test under unobserved heterogeneity.

We conclude this subsection by summarizing the  $M(b, x, q)$  and  $W(b, x, q)$  in the forms of unconditional means in each example in Section 3.

**1. First-price Sealed-bid Auction Model.** For example, in the context of timber

auctions, a natural auction-level covariate is tract volume, which scales the economic stakes of the sale and can mechanically affect bids even in the absence of changes in strategic incentives.

$$\begin{aligned}
M(b, x, q) &= E \left[ B_{i,\ell} 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right] \\
&\quad + \frac{1}{N-1} E \left[ \left( 1 \left( B_{i,\ell} \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - B_{i,\ell} \right) - 1 \left( B_{i,\ell} \leq b \right) (b - B_{i,\ell}) \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right], \\
W(b, x, q) &= E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right].
\end{aligned}$$

For procurement auctions as in Remark 3.2, a natural auction-level covariate is engineer's estimate, as in the empirical application in this paper. We have

$$\begin{aligned}
M(b, x, q) &= E \left[ B_{i,\ell} 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right] - \frac{1}{N-1} \frac{a}{q} E \left[ 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right] \\
&\quad + \frac{1}{N-1} E \left[ \left( 1 \left( B_{i,\ell} \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - B_{i,\ell} \right) - 1 \left( B_{i,\ell} \leq b \right) (b - B_{i,\ell}) \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right], \\
W(b, x, q) &= E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right].
\end{aligned}$$

2. **Tullock Contest Model.** In a Tullock-type contests, prize shifters are one of the key covariates because they scale the marginal benefit of effort. The optimal effort responds mechanically to the prize, i.e., higher stakes raise equilibrium effort and change the distribution of observed actions.

$$\begin{aligned}
M(b, x, q) &= E \left[ B_{i,\ell} \cdot 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right], \\
W(b, x, q) &= E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \frac{B_{i,\ell}}{\sum_{j=1}^N B_{j,\ell}} \left( 1 - \frac{B_{i,\ell}}{\sum_{j=1}^N B_{j,\ell}} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right].
\end{aligned}$$

3. **Public Good Provision Model.** Let  $\Omega(\cdot, x)$  be a known common benefit function of the total contributions conditional on  $X = x$ . A salient covariate in models of public good provision is the endowment. In experimental settings, the endowment is a core game-level primitive because it shapes both the maximum amount that can be allocated to the public good and the monetary payoffs associated with any given

contribution profile.

$$M(b, x, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right],$$

$$W(b, x, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \Omega' \left( \sum_{j=1}^N B_{j,\ell}, X_\ell \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right].$$

4. **Cournot Competition Model.** Let  $\mathcal{I}(\cdot, x)$  be the inverse demand function conditional on  $X = x$  which is known. Market level covariates include demand shifters such as population, income, and seasonality. It is important to control demand shifters in demand estimation because these variables shift the level (and often the slope) of market demand, and therefore move equilibrium prices and quantities even when firms' costs and conduct are unchanged. Ignoring these covariates results in lack of comparability across markets or time, and can lead the econometrician to misattribute demand-driven variation in outputs and prices to differences in marginal costs, productivity, or the degree of market power.

$$M(b, x, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right],$$

$$W(b, x, q) = E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \left( \mathcal{I} \left( \sum_{j=1}^N B_{j,\ell}, X_\ell \right) \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right]$$

$$+ E \left[ 1 \left( b \leq B_{i,\ell} \leq b + \frac{a}{q} \right) \left( \mathcal{I}' \left( \sum_{j=1}^N B_{j,\ell}, X_\ell \right) B_{i,\ell} \right) 1 \left( x \leq X_\ell \leq x + \frac{1}{q} \right) \right].$$

## 4.2 Observed Game Heterogeneity: Semiparametric Test

In Section 4.1, we consider a nonparametric test for games of incomplete information covariates. While it is general, it may suffer from the curse of dimensionality as is usually the case with nonparametric estimation/testing, when the number of (continuous) covariates is large. In this section, we follow Haile et al. (2003), Ma et al. (2021) and Compiani et al. (2020) and impose a semiparametric model to homogenize the actions. Specifically, let  $\Gamma(x)$  be a strictly positive function of covariates. The rest of the framework follows Section 4.1.

**Assumption 4.2** Assume that (i)  $\{(V_{1,\ell} = \Gamma(X_\ell)u_{1,\ell}, V_{2,\ell} = \Gamma(X_\ell)u_{2,\ell}, \dots, V_{N,\ell} = \Gamma(X_\ell)u_{N,\ell}, X_\ell) : \ell = 1, \dots, L\}$  are i.i.d. random vectors; (ii)  $\{u_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$  are i.i.d. random variables with PDF  $f_u(\cdot)$  and CDF  $F_u(\cdot)$  and are independent of  $X_\ell$ . (iii)  $f_u(\cdot)$  is strictly positive and bounded away from zero on its support, a compact interval  $[\underline{u}, \bar{u}] \subseteq R$ , and is twice continuously differentiable on  $(\underline{u}, \bar{u})$ .

**Assumption 4.3** Assume that  $D_{i,\ell}(B_{1,\ell}, \dots, B_{N,\ell})$  is a homogeneous function of order zero and  $T_{i,\ell}(B_{1,\ell}, \dots, B_{N,\ell})$  is a homogeneous function of order one in that for any  $r > 0$

$$\begin{aligned} D_{i,\ell}(rB_{1,\ell}, \dots, rB_{N,\ell}) &= D_{i,\ell}(B_{1,\ell}, \dots, B_{N,\ell}), \\ T_{i,\ell}(rB_{1,\ell}, \dots, rB_{N,\ell}) &= rT_{i,\ell}(B_{1,\ell}, \dots, B_{N,\ell}). \end{aligned}$$

Let  $S_v(v|x)$  denote the BNE strategy. Let  $B_i^u$  denote the actions when agent  $i$ 's value is  $u_i$ . We claim that  $S_v(v|x) = \Gamma(x)s_u(\Gamma^{-1}(x)v)$  for all  $x \in \mathcal{X}$ . That is, if Assumption 4.3 holds, we want to show that “ $b^u \in s(u)$ ”  $\Leftrightarrow$  “ $b = \Gamma(x)b^u \in s_v(v|x) = s_v(\Gamma(x)u|x)$ ”.

Let  $P_{i,\ell}^u(b^u)$  and  $T_{i,\ell}^u(b^u)$  be the expected allocation and transfer that agent  $i$  will attain if agent  $i$  plays  $b^u$ . To show “ $\Rightarrow$ ” direction, note that if  $b^u \in s(u)$ , we have

$$\begin{aligned} u_{i,\ell}P_{i,\ell}^u(b^u) - T_{i,\ell}^u(b^u) &\geq u_{i,\ell}P_{i,\ell}^u(b^{u'}) - T_{i,\ell}^u(b^{u'}) \quad \text{for all } b^{u'} \\ \Leftrightarrow \Gamma(x)u_{i,\ell}P_{i,\ell}^u(b^u) - \Gamma(x)T_{i,\ell}^u(b^u) &\geq \Gamma(x)u_{i,\ell}P_{i,\ell}^u(b^{u'}) - \Gamma(x)T_{i,\ell}^u(b^{u'}) \quad \text{for all } b^{u'} \\ \Leftrightarrow V_{i,\ell}P_{i,\ell}^u(\Gamma(x)b^u) - T_{i,\ell}^u(\Gamma(x)b^u) &\geq V_{i,\ell}P_{i,\ell}^u(\Gamma(x)b^{u'}) - T_{i,\ell}^u(\Gamma(x)b^{u'}) \quad \text{for all } b^{u'} \\ \Leftrightarrow V_{i,\ell}P_{i,\ell}(b|x) - T_{i,\ell}(b|x) &\geq V_{i,\ell}P_{i,\ell}(b'|x) - T_{i,\ell}(b'|x) \quad \text{for all } b', \end{aligned}$$

where the first  $\Leftrightarrow$  holds by rescaling everything by  $\Gamma(x)$ , the second one holds by the fact that  $V_{i,\ell} = \Gamma(x)u_{i,\ell}$  and  $P_{i,\ell}^u(\cdot)$  and  $T_{i,\ell}^u(\cdot)$  are homogeneous functions of degree zero and one, respectively. The last  $\Leftrightarrow$  holds because for agent  $\ell$ , if other players have  $\Gamma(x)b^u \in s_v(v|x)$  being their best response functions, then  $P_{i,\ell}(b|x) = P_{i,\ell}^u(\Gamma(x)b^u)$  and  $T_{i,\ell}(b|x) = T_{i,\ell}^u(\Gamma(x)b^u)$ . It follows that  $b = \Gamma(x)b^u$  is agent  $i$ 's best response hetero too. That is,  $b = \Gamma(x)b^u \in s_v(v|x) = s_v(\Gamma(x)u|x)$ . The same arguments apply to the “ $\Leftarrow$ ” direction of the claim.

This result implies that

$$\begin{aligned} \xi(b, x) &\equiv \frac{T'(b, x)}{P'(b, x)} \text{ is strictly increasing in } b \text{ for all } x \in \mathcal{X}. \\ \Leftrightarrow \xi(b^u) &\equiv \frac{T^w(b^u)}{P^w(b^u)} \text{ is strictly increasing in } b^u. \end{aligned} \quad (4.6)$$

(4.6) further implies that we can re-formulate the null hypothesis  $H_{0,x}$  in (4.2) as

$$H_{0,\text{semi}} : \xi(b^u) \equiv \frac{T^w(b^u)}{P^w(b^u)} \text{ is weakly increasing in } b^u. \quad (4.7)$$

Therefore, then  $H_{0,\text{semi}}$  in (4.7) is equivalent to

$$H'_{0,\text{semi}} : \nu(b_1^u, b_2^u, q) = M(b_2^u, q)W(b_1^u, q) - M(b_1^u, q)W(b_2^u, q) \leq 0 \text{ for all } (b_1^u, b_2^u, q) \in \mathcal{L},$$

where

$$\begin{aligned} M(b^u, q) &= \int_{b^u}^{b^u + \frac{a}{q}} h(\tilde{b})T'(\tilde{b})d\tilde{b}, \quad W(b^u, q) = \int_{b^u}^{b^u + \frac{a}{q}} h(\tilde{b})P'(\tilde{b})d\tilde{b}, \text{ and} \\ \mathcal{L} &= \left\{ (b_1^u, b_2^u, q) : \left( \frac{b_1^u - b^u}{a}, \frac{b_2^u - b^u}{a} \right) \cdot q \in (0, 1, \dots, q)^2, b_1^u > b_2^u \text{ for } q=2,3,\dots \right\}. \end{aligned}$$

We note that first, Assumption 4.3 is crucial for the homogenization method to work. Therefore, among the four examples discussed in Section 3, the homogenization method can be applied to only the first-price sealed-bid auction model and Tullock contest model. Second, the intuition of the equivalence result is that when the actions are monetary activities such as the auction games, then if we use different currencies, the resulting bids should be the same in the sense that the resulting bids should have the relation as above.

If  $\Gamma(x)$  is known, then we obtain the rescaled actions,  $B_{i,\ell}^u = \Gamma^{-1}(X_\ell)B_{i,\ell}$  and apply the test with the rescaled actions in Section 2.2 to test  $H_{0,\text{semi}}$  in (4.7). If  $\Gamma(x)$  follows a parametric specification  $\Gamma(x, \theta)$  so that  $\theta$  is identified and a consistent estimator  $\hat{\theta}$  is available, then we can apply a similar test in Section 2.2 after accounting for the estimation effect of  $\hat{\theta}$  in the test. In the Appendix, we present a test in auction games and show its validity under regularity conditions.

### 4.3 Unobserved Game Heterogeneity

We extend our tests in Section 4.1 to allow for unobserved heterogeneity in the model to the nonparametric case. For semiparametric case, please see Appendix E for details.

Suppose that for each game, in addition to an observed vector of covariates,  $X$ , there is an unobserved vector of  $Z$  denoting the unobserved characteristics which is observable or is common knowledge among players, but not directly observable to econometricians. Without loss of generality, we assume both  $X$  and  $Z$  to be scalars with  $\mathcal{X} = [0, 1]$  and  $\mathcal{Z} = [0, 1]$ .<sup>14</sup> The model is the same as in Section 4.1 except that we have an additional unobserved covariate  $Z$ . For a given strategy  $s$  and given  $X = x$  and  $Z = z$ , we write  $s(v, x, z)$  for the action played by type  $v$ .

Following Section 4.1, we would have

$$\xi(b, x, z) \equiv \frac{T'(b, x, z)}{P'(b, x, z)} \text{ is strictly increasing in } b. \quad (4.8)$$

However, we cannot formulate the null hypothesis as in 4.2, because  $Z$  is unobservable to econometricians. Instead, (4.8) implies the following testable implications.

**Lemma 4.2** *Let  $h(b, x, z)$  be a known function such that  $0 < h(b, x, z)P'(b, x, z) \leq M < \infty$ . Then (4.8) implies that*

$$H'_{0,x,z} : \nu(b_1, b_2, x, q) = M(b_2, x, q)W(b_1, x, q) - M(b_1, x, q)W(b_2, x, q) \leq 0$$

for all  $(b_1, b_2, x, q) \in \mathcal{L}_x$ , (4.9)

where

$$M(b, x, q) = \int_0^1 \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{q}{q}} h(\tilde{b}, \tilde{x}, z) T'(\tilde{b}, \tilde{x}, z) d\tilde{b} d\tilde{x} dz$$

$$W(b, x, q) = \int_0^1 \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{q}{q}} h(\tilde{b}, \tilde{x}, z) P'(\tilde{b}, \tilde{x}, z) d\tilde{b} d\tilde{x} dz, \text{ and} \quad (4.10)$$

$$\mathcal{L}_x = \left\{ (b_1, b_2, x, q) : \left( \frac{b_1 - \underline{b}}{a}, \frac{b_2 - \underline{b}}{a}, x \right) q \in (0, 1, \dots, q)^3, b_1 > b_2 \text{ for } q=2, 3, \dots \right\}. \quad (4.11)$$

---

<sup>14</sup>It is straightforward to allow for  $X$  and  $Z$  being vectors of observed and unobserved characteristics, respectively.

The basic idea of Lemma 4.2 is to integrate out  $Z$ , so  $Z$  will not show up in the expressions anymore. In this case, we may use the same tests as in Section 4.1. The  $M(b, x, q)$  and  $W(b, x, q)$  for four examples are identical to those in Section 4.1 and we omit the details. Note that the testable implications in these cases are the same as those in Section 4.1, so we can apply the same test in Section 4.1 to test the models with unobserved heterogeneity and we omit the details. The extension to settings without observed covariates  $X$  follows by a similar argument, so we omit the details.

## 5 Simulation Studies

We conduct Monte-Carlo simulations to examine the finite sample performance of the proposed tests. To implement our test in practice, one has to choose several tuning parameters in advance. We first suggest the choices of these tuning parameters and then present the related Monte Carlo simulation results.

1. Instrumental functions: We opt for using a set of indicator functions of countable hypercubes. Define

$$\mathcal{L} = \{(b_1, b_2, x, q) : (b_1, b_2, x)q \in \{0, 1, \dots, q-1\}^{2+d_x}, b_1 > b_2 \text{ for } q = 2, 3, \dots, q_1\},$$

where  $q_1$  is a natural number and is chosen such that the expected sample size of the smallest cube is around 20. We also consider three alternative choices of  $q_1$ , each resulting in the expected sample size of the smallest cube  $n_c = 15, 25$ , and  $30$ .<sup>15</sup> Our simulations show that the results are robust to various expected sample sizes.

2.  $Q(b_1, b_2, x, q)$ : The distribution  $Q(b_1, b_2, x, q)$  assigns weight  $\propto q^{-2}$  to each  $q$ , and for each  $q$ ,  $Q(\ell)$  assigns an equal weight to each instrumental function with the last element of  $\ell$  equal to  $q^{-1}$ . Recall that for each  $q$ , there are  $(q(q+1)/2) \cdot q^{d_x}$  instrumental functions with the last element of  $(b_1, b_2, x, q)$  equal to  $q$ .

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<sup>15</sup>To be specific, for a specific  $n_c$ , we set  $q_1 = \lfloor (S/n_c)^{1/(1+d_x)} \rfloor$ , where  $\lfloor c \rfloor$  gives the integer nearest to  $c$  and  $d_x$  is the dimension of (continuous) covariates. In addition, for heterogeneous numbers of bidders, we have  $q_{1,t}$  a function of  $t$  and  $q_{1,t} = \lfloor (S_t/n_c)^{1/(1+d_x)} \rfloor$ .

3.  $\kappa_S, \beta_S, \epsilon, \eta$ : We set  $\kappa_S = 0.15 \cdot \ln(S)$ ,  $\beta_S = 0.85 \cdot \ln(S) / \ln \ln(S)$ ,  $\epsilon = 10^{-6}$ , and  $\eta = 10^{-6}$  as suggested by Hsu et al. (2019). These choices are used in all the simulations that we report below and seem to perform well. For the case with heterogeneous number of bidders, we set  $\kappa_{S_t} = 0.15 \cdot \ln(S_t)$ ,  $\beta_{S_t} = 0.85 \cdot \ln(S_t) / \ln \ln(S_t)$ .
4. All our Monte Carlo results are based on 1000 simulations. In each simulation, the critical value is approximated by 1000 bootstrap replications. The nominal size of the test is set at 10%.

We first consider the case without covariates. To study the size and power properties, the CDF of the observed bids is specified as

$$G(b) = \left( \frac{b}{k - (k-1)b} \right)^{1/5}, \quad (5.1)$$

and the associated quantile function of the true bid distribution is

$$Q(\tau) = \frac{k\tau^5}{1 + (k-1)\tau^5}. \quad (5.2)$$

We set  $k = 0.5$  for the size analysis and set  $k = 5, 10, 15$ , and  $20$  for the power analysis. Note that  $k = 5$  corresponds to the example that violates the monotonicity condition provided in GPV. For all values of  $k$ , we set  $N = 2$  and  $L = 100, 250$ , or  $500$ . Figure 1 presents the quasi-inverse equilibrium strategy,  $\xi(b) = b + G(b)/(g(b)(N-1))$  for different values of  $k$ . From this figure, we can observe that the monotonicity condition holds for  $k = 0.5$ , but is violated for  $k \geq 5$ . The figure also shows that it will be easier to detect the violation as  $k$  increases.

Table 1 shows the rejection probabilities of our tests. The results show that the proposed test controls size well for  $k = 0.5$ . Note that when  $k = 0.5$ , the null hypothesis holds with strict inequality. Therefore, the size will converge to zero since every moment would hold with strict inequality. It follows that the test statistics will converge to zero and the critical value is bounded away from zero. For the power analysis, the results show that the power increases with sample size and with  $k$  for  $k \geq 5$ , but the rejection probabilities are a bit less than the nominal size 0.1 for  $k = 5$ . Table 1 also shows that the choices of  $q_1$  do not affect

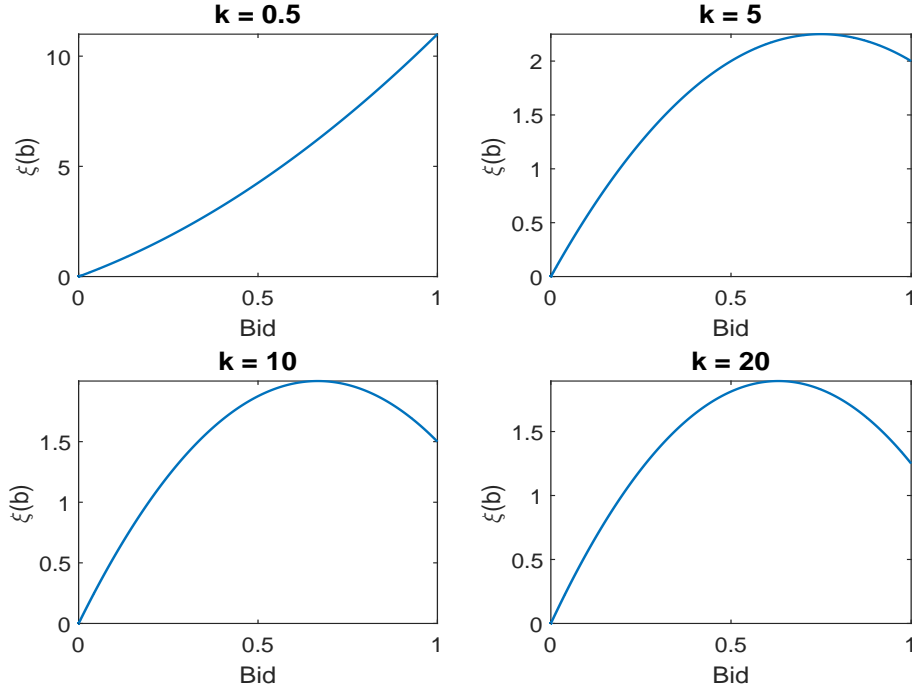


Figure 1: Quasi-inverse equilibrium strategy,  $\xi(b)$

the test performance much.

Table 1: Rejection probabilities for the case without covariates

k	L	$n_c=15$	$n_c=20$	$n_c=25$	$n_c=30$
0.5	100	0.001	0.004	0.001	0.002
0.5	250	0.003	0.003	0.001	0.003
0.5	500	0.001	0.002	0.001	0.001
5	100	0.080	0.084	0.084	0.067
5	250	0.080	0.070	0.073	0.077
5	500	0.084	0.084	0.098	0.106
10	100	0.311	0.310	0.297	0.297
10	250	0.504	0.519	0.470	0.464
10	500	0.737	0.754	0.741	0.741
20	100	0.710	0.690	0.700	0.682
20	250	0.953	0.939	0.958	0.951
20	500	1.000	1.000	1.000	1.000

We next consider the case with covariates. Let  $X$  to be a uniform distribution on  $[0, 1]$ . To consider the test with covariates, the quantile function of the true bid distribution is

specified as

$$Q(x, \tau) = \frac{k(x)\tau^5}{1 + (k(x) - 1)\tau^5}.$$

We consider the following cases for the size and power analyses:

- Case 1:  $k(x) = 0.5 + 2x$  for the size case,
- Case 2:  $k(x) = 5 + 5x$  for the power case,
- Case 3:  $k(x) = 10 + 5x$  for the power case,
- Case 4:  $k(x) = 15 + 5x$  for the power case,
- Case 5:  $k(x) = 20 + 5x$  for the power case.

For all cases, we set  $N = 2$  and  $L = 100, 250, 500,$  or  $1000$ . Table 2 shows the rejection probabilities of our tests for the case with covariates, and the results are consistent with our theoretical findings. The proposed test controls size well in Case 1, and the rejection probabilities increase with the sample size and are greater than the nominal size 0.1 in Cases 2-5. Like the results in Table 1, we do not find significant differences for different choices  $q_1$ .

Table 2: Rejection probabilities for the case with covariates

Case	L	$n_c=15$	$n_c=20$	$n_c=25$	$n_c=30$
1	100	0.013	0.019	0.014	0.021
1	250	0.002	0.004	0.004	0.007
1	500	0.003	0.000	0.002	0.001
1	1000	0.000	0.000	0.000	0.000
2	100	0.119	0.128	0.137	0.136
2	250	0.123	0.143	0.144	0.127
2	500	0.175	0.172	0.155	0.165
2	1000	0.256	0.256	0.240	0.248
3	100	0.234	0.237	0.225	0.231
3	250	0.412	0.413	0.391	0.351
3	500	0.648	0.643	0.622	0.620
3	1000	0.910	0.896	0.910	0.885
4	100	0.364	0.313	0.288	0.325
4	250	0.663	0.638	0.596	0.606
4	500	0.927	0.920	0.910	0.902
4	1000	0.998	0.997	0.997	0.999
5	100	0.463	0.412	0.409	0.419
5	250	0.797	0.794	0.760	0.760
5	500	0.978	0.985	0.972	0.985
5	1000	1.000	1.000	1.000	1.000

## 6 Empirical Application

As discussed in Bajari and Ye (2003), monotonicity of equilibrium bidding strategies is one of the sufficient and necessary conditions for the symmetric and competitive bidding equilibrium. While they try to test for efficient collusion by testing these conditions, they do not test for monotonicity due to the lack of monotonicity tests.<sup>16</sup> Clearly, rejection of monotonicity can be indicative of potential collusive behavior. To demonstrate the effectiveness of our monotonicity test, we use the asphalt paving auction datasets studied in Aaltio et al. (2025), who analyze bidding behavior in state-level road-paving procurements in Sweden, Finland and California.

<sup>16</sup>Aryal and Gabrielli (2013) propose a stochastic dominance test to test for collusion. To implement the test, they first use the tests of Bajari and Ye (2003) to identify potential colluding bidders.

## 6.1 Data

The Nordic data are drawn from contracts procured by the Swedish Road Administration and the Finnish Transport Infrastructure Agency, covering 1993-2009 in Sweden and 1994-2019 in Finland. Both datasets contain all submitted bids in each tender, the identity of the winning bidder, paving area ( $m^2$ ), and the region where the pavement project took place.

Both Nordic markets experienced bid-rigging cartels during the late 1990s and early 2000s. To focus on bidding behavior during legally established cartel operation while avoiding contamination around detection and enforcement, we restrict attention to the court-documented conviction windows and exclude the dawn-raid years. Specifically, we use tenders from 1997–2000 in Sweden and 1995–2001 in Finland.<sup>17</sup> The California asphalt paving market is widely considered as competitive and there is no evidence of collusion during this period, and used in Aaltio et al. (2025) as a control market. We use California asphalt paving contracts procured by the California Department of Transportation from 1999 to 2008.

For the monotonicity test, we work with normalized bids to make outcomes comparable across contracts of different scales. In Sweden and Finland, we normalize bids by paving area (price per  $m^2$ ). In the test that control for observed auction heterogeneity, we control for paving area. The California dataset contains the similar information as the Nordic datasets, including all submitted bids, winner identity, and project region, except that the former contains the information on engineer’s cost estimate but not paving area, while the latter contains the information on paving area but not engineer’s cost estimate. In California, we normalize bids by engineer’s cost estimate, which is also controlled for in the test that controls for observed auction heterogeneity.

## 6.2 Test results

We first conduct the monotonicity test for the California dataset. Table 3 reports the test statistics and associated p-values for auctions with 2 to 10 bidders and heterogeneous numbers of bidders. The tuning parameters are chosen as in Section 5, and the test with heterogeneous numbers of bidders is discussed in Appendix B. The left panel of Table 3 reports the

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<sup>17</sup>Both markets featured complementary bidding consistent with designated-winner coordination.

test results based on the nonparametric test without controlling for the covariates, and the right panel of Table 3 reports the test results based on the nonparametric test with covariates discussed in Section 4.1. The results show that there is no case with strong evidence against the monotonicity null hypothesis, which is consistent with the literature where the California market is usually modeled as competitive and no cartel investigation was conducted during 1999 to 2008.

We next apply the monotonicity tests to the Sweden and Finland datasets. Tables 4 and 5 report the test statistics and associated p-values for auctions with 3 to 8 bidders and heterogeneous numbers of bidders for Sweden and Finland, respectively. For Sweden, the p-values for the auction with 5 bidders and the cases associated with 5 bidders are less than 0.1, which provide strong evidence against the monotonicity. For Finland, the p-values for some cases in nonparametric tests with covariates are around 0.3, which provide weak evidence against the monotonicity. Interestingly, Aaltio et al. (2025) find that the cartel had a larger impact on the distribution of bids in Finland than in Sweden, and they interpret this as that the Swedish cartel was more successful in mimicking competitive behavior, or it had more limited influence in scope. The strong evidence against monotonicity we find for the Swedish market shows the power of our test in detecting non-competitive behavior which other methods may not be able to detect.

Table 3: Test statistics and p-values for California

$N_t$	without covariates				with covariates			
	$n_c=15$		$n_c=20$		$n_c=15$		$n_c=20$	
	stat	p-value	stat	p-value	stat	p-value	stat	p-value
2	0.0318	0.3268	0.0233	0.3878	0.0001	0.6566	0.0001	0.6564
(2,3)	0.0414	0.6710	0.0322	0.7170	0.0001	0.9968	0.0001	0.9846
(2,3,4)	0.1306	0.7114	0.1210	0.7270	0.0122	0.8146	0.0111	0.7870
3	0.0096	0.6790	0.0089	0.6856	0.0000	1.0000	0.0000	1.0000
(3,4)	0.0987	0.6844	0.0977	0.6794	0.0121	0.7360	0.0110	0.7140
(3,4,5)	0.1165	0.8178	0.1153	0.8114	0.0128	0.8934	0.0115	0.8772
4	0.0892	0.5566	0.0888	0.5432	0.0121	0.5784	0.0110	0.5312
(4,5)	0.1069	0.7298	0.1064	0.7310	0.0128	0.7786	0.0115	0.7484
(4,5,6)	0.1512	0.8396	0.1506	0.8346	0.0172	0.8930	0.0153	0.8834
5	0.0178	0.6782	0.0176	0.6676	0.0007	0.9360	0.0005	0.9416
(5,6)	0.0620	0.7662	0.0619	0.7728	0.0051	0.9074	0.0043	0.9216
(5,6,7)	0.0626	0.8700	0.0624	0.8604	0.0051	0.9892	0.0043	0.9878
6	0.0442	0.6314	0.0442	0.6254	0.0044	0.7738	0.0038	0.7734
(6,7)	0.0449	0.7794	0.0448	0.7912	0.0044	0.9524	0.0038	0.9518
(6,7,8)	0.0489	0.8718	0.0470	0.8662	0.0044	0.9826	0.0038	0.9840
7	0.0006	0.8814	0.0006	0.8270	0.0000	1.0000	0.0000	1.0000
(7,8)	0.0047	0.7964	0.0028	0.8548	0.0000	1.0000	0.0000	1.0000
(7,8,9)	0.0562	0.7548	0.0508	0.7644	0.0000	1.0000	0.0000	1.0000
8	0.0041	0.5774	0.0022	0.6120	0.0000	1.0000	0.0000	1.0000
(8,9)	0.0555	0.5884	0.0502	0.6024	0.0000	1.0000	0.0000	1.0000
(8,9,10)	0.0586	0.7542	0.0530	0.7512	0.0000	1.0000	0.0000	1.0000

Note: ‘Stat’ denotes the test statistic, and the p-values of the tests are calculated based on 5000 bootstrap replications. The number of bidders  $N_t$  and the number of auctions  $L_t$  are:  $\{(N_t, L_t) : (2, 204), (3, 308), (4, 276), (5, 212), (6, 163), (7, 117), (8, 58), (9, 41), (10, 33)\}$ .

Table 4: Test statistics and p-values for Sweden

$N_t$	without covariates				with covariates			
	$n_c=15$		$n_c=20$		$n_c=15$		$n_c=20$	
	stat	p-value	stat	p-value	stat	p-value	stat	p-value
3	0.1381	0.0744	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
(3,4)	0.2589	0.2034	0.1050	0.2434	0.0000	1.0000	0.0000	1.0000
(3,4,5)	0.9441	0.0508	0.7738	0.0636	0.0823	0.0698	0.0630	0.0766
4	0.1208	0.2340	0.1050	0.2316	0.0000	1.0000	0.0000	1.0000
(4,5)	0.8060	0.0616	0.7738	0.0636	0.0823	0.0688	0.0630	0.0874
(4,5,6)	0.8907	0.0530	0.8360	0.0484	0.0823	0.0590	0.0630	0.0768
5	0.6852	0.0438	0.6688	0.0378	0.0823	0.0836	0.0630	0.0866
(5,6)	0.7699	0.0346	0.7311	0.0352	0.0823	0.0838	0.0630	0.0786
(5,6,7)	1.4276	0.0722	1.3355	0.0796	0.1719	0.1198	0.0630	0.0842
6	0.0847	0.1866	0.0622	0.2058	0.0000	1.0000	0.0000	1.0000
(6,7)	0.7424	0.0982	0.6667	0.0786	0.0896	0.1372	0.0000	1.0000

Note: ‘Stat’ denotes the test statistic, and the p-values of the tests are calculated based on 5000 bootstrap replications. The number of bidders  $N_t$  and the number of auctions  $L_t$  are:  $\{(N_t, L_t) : (3, 15), (4, 58), (5, 67), (6, 28), (7, 17)\}$ .

Table 5: Test statistics and p-values for Finland

$N_t$	without covariates				with covariates			
	$n_c=15$		$n_c=20$		$n_c=15$		$n_c=20$	
	stat	p-value	stat	p-value	stat	p-value	stat	p-value
3	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
(3,4)	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
(3,4,5)	0.0474	0.7502	0.0293	0.7620	0.0374	0.3484	0.0000	1.0000
4	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
(4,5)	0.0474	0.7260	0.0293	0.7446	0.0374	0.3568	0.0000	1.0000
(4,5,6)	0.1595	0.6678	0.1217	0.6842	0.0374	0.3480	0.0000	1.0000
5	0.0474	0.6766	0.0293	0.7068	0.0374	0.3318	0.0000	1.0000
(5,6)	0.1595	0.6414	0.1217	0.6506	0.0374	0.3208	0.0000	1.0000
(5,6,7)	0.1597	0.8132	0.1217	0.8188	0.0374	0.6020	0.0000	1.0000
6	0.1121	0.4546	0.0924	0.4774	0.0000	1.0000	0.0000	1.0000
(6,7)	0.1124	0.7352	0.0924	0.7092	0.0000	1.0000	0.0000	1.0000
(6,7,8)	0.1124	0.8434	0.0924	0.8402	0.0000	1.0000	0.0000	1.0000
7	0.0002	0.9390	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000
(7,8)	0.0002	0.9898	0.0000	1.0000	0.0000	1.0000	0.0000	1.0000

Note: ‘Stat’ denotes the test statistic, and the p-values of the tests are calculated based on 5000 bootstrap replications. The number of bidders  $N_t$  and the number of auctions  $L_t$  are:  $\{(N_t, L_t) : (3, 15), (4, 14), (5, 41), (6, 29), (7, 15), (8, 12)\}$ .

## 7 Conclusion

This paper develops a unified framework for testing whether Bayesian Nash equilibrium strategies are monotone in unobserved types in games of incomplete information. We first establish a rationalization result showing that, within the symmetric independent private type paradigm, monotonicity of the differentiable equilibrium strategy in private types is equivalent to monotonicity of the quasi-inverse of the equilibrium strategy in actions; the quasi-inverse equilibrium strategy is a function of expected allocation and expected payment rules, and can be identified from observed actions. As a result, our test reduces to testing monotonicity of the quasi-inverse equilibrium strategy, which can be reformulated as testing a countable collection of moment inequalities that involve only unconditional expectations. The proposed testing procedure is easy to implement, and can accommodate game specific covariates and applies to both observed and unobserved game heterogeneity. The tests have good finite sample performance as demonstrated by the Monte-Carlo experiments. An application to asphalt paving auctions studied in Aaltio et al. (2025) illustrates how the test can be used for cartel detection, and demonstrates the importance of testing for monotonicity and potential applications of our tests. For instance, our tests provide a key tool as a first step in detecting non-competitive behavior.

## References

- Aaltio, A., R. Buri, A. Jokelainen, and J. Lundberg (2025). Complementary bidding and cartel detection: Evidence from nordic asphalt markets. *International Journal of Industrial Organization* 98, 103129.
- Agarwal, N., P. Z. Li, and P. Somaini (2023). Identification using revealed preferences in linearly separable models. Working paper.
- Aguirregabiria, V. and P. Mira (2019). Identification of games of incomplete information with multiple equilibria and unobserved heterogeneity. *Quantitative Economics* 10, 1659–1701.
- Andrews, D. W. K. and X. Shi (2013). Inference based on conditional moment inequalities. *Econometrica* 81(2), 609–666.

- Andrews, D. W. K. and G. Soares (2010). Inference for parameters defined by moment inequalities using generalized moment selection. *Econometrica* 78(2), 119–157.
- Aradillas-Lopze, A. (2010). Semiparametric estimation of a simultaneous games with incomplete information. *Journal of Econometrics* 157, 409–431.
- Aryal, G. and M. F. Gabrielli (2013). Testing for collusion in asymmetric first-price auctions. *International Journal of Industrial Organization* 31(1), 26–35.
- Aryal, G. and M. F. Gabrielli (2020). An empirical analysis of competitive nonlinear pricing. *International Journal of Industrial Organization* 68, 102538.
- Aryal, G. and F. Zincenko (2024). Empirical framework for cournot oligopoly with private information. *RAND Journal of Economics* 55(3), 375–402.
- Athey, S. (2001). Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica* 69, 861–889.
- Bag, P. K. and S. Roy (2011). On sequential and simultaneous contributions under incomplete information. *International Journal of Game Theory* 40(1), 119–145.
- Bajari, P. and L. Ye (2003). Deciding between competition and collusion. *The Review of Economics and Statistics* 85(4), 971–989.
- Campo, S., I. Perrigne, and Q. Vuong (2003). Asymmetry in first-price auctions with affiliated private values. *Journal of Applied Econometrics* 18, 179–207.
- Chetverikov, D., A. Santos, and A. M. Shaikh (2018). The econometrics of shape restrictions. *Annual Review of Economics* 10, 31–63.
- Compiani, G., P. Haile, and M. Sant’Anna (2020). Common values, unobserved heterogeneity, and endogenous entry in us offshore oil lease auctions. *Journal of Political Economy* 128, 3872–3912.
- de Paula, A. and X. Tang (2012). Inference of signs of interaction effects in simultaneous games with incomplete information. *Econometrica* 80, 143–172.
- Donald, S. G. and Y.-C. Hsu (2016). Improving the power of tests of stochastic dominance. *Econometric Reviews* 35, 553–585.
- Ewerhart, C. (2014). Unique equilibrium in rent-seeking contests with a continuum of types. *Economics Letters* 125, 115–118.
- Ewerhart, C. and F. Quartieri (2020). Unique equilibrium in contests with incomplete information. *Economic Theory* 70(1), 243–271.
- Fang, Z. and J. Seo (2021). A projection framework for testing shape restrictions that form convex cones. *Econometrica* 89, 2439–2458.
- Guerre, E., I. Perrigne, and Q. Vuong (2000). Optimal nonparametric estimation of first-price auctions. *Econometrica* 68(3), 525–574.
- Guerre, E., I. Perrigne, and Q. Vuong (2009). Nonparametric identification of risk aversion in first-price auctions under exclusion restrictions. *Econometrica* 77(4), 1193–1227.

- Haile, P. A., H. Hong, and M. Shum (2003). Nonparametric tests for common values at first-price sealed-bid auctions. Working paper.
- Haile, P. A. and Y. Kitamura (2019). Unobserved heterogeneity in auctions. *The Econometrics Journal* 22, C1–C19.
- Hansen, P. R. (2005). A test for superior predictive ability. *Journal of Business and Economics Statistics* 23, 365–380.
- He, M. and Y. Huang (2021). Structural analysis of tullock contests with an application to u.s. house of representatives elections. *International Economic Review* 62, 1011–1054.
- Henderson, D. J., J. A. List, D. L. Daniel L. Millimet, C. F. Parmeter, and M. K. Price (2012). Empirical implementation of nonparametric first-price auction models. *Journal of Econometrics* 168, 17–28.
- Hsu, Y.-C., M. Huber, Y.-Y. Lee, and C.-A. Liu (2025). Testing monotonicity of mean potential outcomes in a continuous treatment with high-dimensional data. *The Review of Economics and Statistics*, forthcoming.
- Hsu, Y.-C., C.-A. Liu, and X. Shi (2019). Testing generalized regression monotonicity. *Econometric Theory* 154, 1146–1200.
- Hsu, Y.-C. and S. Shen (2021). Testing monotonicity of conditional treatment effects under regression discontinuity designs. *Journal of Applied Econometrics* 36, 271–366.
- Kline, B. (2025). Identification of incomplete information games in monotone equilibrium. Working Paper.
- Krasnokutskaya, E. (2011). Identification and estimation of auction models with unobserved heterogeneity. *The Review of Economic Studies* 78, 293–327.
- Larsen, B. J. and A. L. Zhang (2018). A mechanism design approach to identification and estimation. NBER Working Paper 24837.
- Li, T., I. Perrigne, and Q. Vuong (2002). Structural estimation of the affiliated private value auction model. *The RAND Journal of Economics* 33, 171–193.
- Li, T., J. Zhang, and J. Zhao (2025). Nonparametric identification of games of incomplete information. Working paper.
- Li, T., J. Zhang, and J. Zhao (2026). Nonparametric identification and estimation for Bayesian games. Working paper.
- Li, T. and X. Zheng (2009). Entry and competition effects in first-price auctions: Theory and evidence from procurement auctions. *The Review of Economic Studies* 76, 1397–1429.
- Linton, O., K. Song, and Y.-J. Whang (2010). An improved bootstrap test of stochastic dominance. *Journal of Econometrics* 154, 186–202.
- Liu, N. and Q. Vuong (2021). Nonparametric tests for monotonicity of strategies in games of incomplete information. Working paper.
- Luo, Y., P. Sang, and R. Xiao (2024). Order statistics approaches to unobserved heterogeneity in auctions. *Electronic Journal of Statistics* 18, 2477–2530.

- Ma, J., V. Marmer, A. Shneyerov, and P. Xu (2021). Monotonicity-constrained nonparametric estimation and inference for first-price auctions. *Econometric Reviews* 40, 944–982.
- McAdams, D. (2003). Isotone equilibrium in games of incomplete information. *Econometrica* 71, 1191–1214.
- Reny, P. J. (2011). On the existence of monotone pure strategy equilibria in Bayesian games. *Econometrica* 79, 499–533.
- Roberts, J. W. (2013). Unobserved heterogeneity and reserve prices in auctions. *The Review of Economic Studies* 44, 712–732.
- Romano, J. P. (2004). On non-parametric testing, the uniform behaviour of the t-test, and related problems. *Scandinavian Journal of Statistics* 31, 567–584.
- Ryvkin, D. (2010). Contests with private costs: Beyond two players. *European Journal of Political Economy* 26, 558–567.
- Seim, K. (2006). An empirical model of firm entry with endogenous product-type choices. *RAND Journal of Economics* 37(3), 619–640.
- Tang, X. (2010). Estimating simultaneous games with incomplete information under median restrictions. *Economics Letters* 108, 273–276.
- Tullock, G. (1980). Efficient rent seeking. In J. Buchanan, G. Tullock, and R. Tollison (Eds.), *Toward a Theory of the Rent-Seeking Society (Texas A & M University Economics Series)*, pp. 97–112. Texas A & M University Press.
- van der Vaart, A. W. and J. A. Wellner (1996). *Weak Convergence and Empirical Processes with Applications to Statistics*. Springer Series in Statistics. Springer.
- Varian, H. R. (1994). Sequential contributions to public goods. *Journal of Public Economics* 53(2), 165–186.
- Vives, X. (2002). Private information, strategic behavior, and efficiency in cournot markets. *RAND Journal of Economics* 33(3), 361–376.
- Wan, Y. and H. Xu (2014). Semiparametric identification of binary decision games of incomplete information with correlated private signals. *Journal of Econometrics* 182, 235–246.

# Appendix

## A Details of the Test in Auction Games in Section 3.1

Given that our main simulation results and empirical studies focused on the auction case, we provide the details of the test and provide an implementation procedure. Based on the identification results (3.5) and (3.6) in Lemma 3.1, we can estimate  $\nu(b_1, b_2, q)$  by

$$\begin{aligned}\widehat{\nu}(b_1, b_2, q) &= \widehat{M}(b_2, q)\widehat{W}(b_1, q) - \widehat{M}(b_1, q)\widehat{W}(b_2, q), \text{ where} & (A.1) \\ \widehat{M}(b, q) &= \frac{1}{S} \sum_{i,\ell} \left( B_{i,\ell} 1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right) \right. \\ &\quad \left. + \frac{1}{N-1} \left( 1\left(B_{i,\ell} \leq b + \frac{a}{q}\right) \left(b + \frac{a}{q} - B_{i,\ell}\right) - 1(B_{i,\ell} \leq b)(b - B_{i,\ell}) \right) \right), \\ \widehat{W}(b, q) &= \frac{1}{S} \sum_{i,\ell} 1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right).\end{aligned}$$

In Appendix, we show that

$$\begin{aligned}\phi_{i,\ell,M}(b, q) &= \left( B_{i,\ell} 1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right) \right. \\ &\quad \left. + \frac{1}{N-1} \left( 1\left(B_{i,\ell} \leq b + \frac{a}{q}\right) \left(b + \frac{a}{q} - B_{i,\ell}\right) - 1(B_{i,\ell} \leq b)(b - B_{i,\ell}) \right) \right) - M(b, q), \\ \phi_{i,\ell,W}(b, q) &= 1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right) - W(b, q), \\ \phi_{i,\ell,\nu}(b_1, b_2, q) &= W(b_1, q)\phi_{i,\ell,M}(b_2, q) + M(b_2, q)\phi_{i,\ell,W}(b_1, q) \\ &\quad - W(b_2, q)\phi_{i,\ell,M}(b_1, q) - M(b_1, q)\phi_{i,\ell,W}(b_2, q),\end{aligned}$$

are the influence functions for estimators  $\widehat{M}(b, q)$ ,  $\widehat{W}(b, q)$  and  $\widehat{\nu}(b_1, b_2, q)$ , respectively, so that

$$\sqrt{S}(\widehat{\nu}(\cdot, \cdot, \cdot) - \nu(\cdot, \cdot, \cdot)) \Rightarrow \Phi(\cdot, \cdot, \cdot)$$

in which  $\Phi(\cdot, \cdot, \cdot)$  is a Gaussian process with covariance kernel being  $\kappa((b'_1, b'_2, q'), (b''_1, b''_2, q'')) = Cov(\phi_{i,\ell,\nu}(b'_1, b'_2, q'), \phi_{i,\ell,\nu}(b''_1, b''_2, q''))$ .

Define  $\widehat{\phi}_{i,\ell,M}(b, q)$ ,  $\widehat{\phi}_{i,\ell,W}(b, q)$  and  $\widehat{\phi}_{i,\ell,\nu}(b_1, b_2, q)$  by replacing  $M(b, q)$  and  $W(b, q)$  with  $\widehat{M}(b, q)$  and  $\widehat{W}(b, q)$  in  $\phi_{i,\ell,M}(b, q)$ ,  $\phi_{i,\ell,W}(b, q)$  and  $\phi_{i,\ell,\nu}(b_1, b_2, q)$ , respectively. Define  $\sigma_\nu^2(b_1, b_2, q) = \kappa((b_1, b_2, q), (b_1, b_2, q))$  which is the asymptotic variance of  $\sqrt{S}(\widehat{\nu}(b_1, b_2, q) - \nu(b_1, b_2, q))$  and its estimator as  $\widehat{\sigma}_\nu^2(b_1, b_2, q)$

$$\widehat{\sigma}_\nu^2(b_1, b_2, q) = \frac{1}{S} \sum_{i,\ell} \widehat{\phi}_{i,\ell,\nu}^2(b_1, b_2, q),$$

which can be shown to be uniformly consistent for  $\sigma_\nu^2(b_1, b_2, q)$ . Define

$$\widehat{\sigma}_{\nu,\epsilon}^2(b_1, b_2, q) = \max\{\widehat{\sigma}_\nu^2(b_1, b_2, q), \epsilon \cdot \widehat{\sigma}_\nu^2(\underline{b}, (\underline{b} + \bar{b})/2, 2)\}. \quad (\text{A.2})$$

The test statistic is defined as

$$\widehat{T}_{S,au} = \sum_{(b_1, b_2, q) \in \mathcal{L}} \max\left\{\sqrt{S} \frac{\widehat{\nu}(b_1, b_2, q)}{\widehat{\sigma}_{\nu,\epsilon}(b_1, b_2, q)}, 0\right\}^2 Q(b_1, b_2, q). \quad (\text{A.3})$$

For the nonparametric bootstrap, we bootstrap auctions. That is, let  $\{(B_{1,\ell}^*, \dots, B_{N\ell}^*) : \ell \leq L\}$  be a bootstrap sample in which  $B_{i,\ell}^* = B_{i\ell^*}$  and  $\{\ell^* : \ell \leq L\}$  is an i.i.d. bootstrap sample drawn from the empirical distribution of  $\{\ell : \ell \leq L\}$ . Define the bootstrap estimator for  $\nu(b_1, b_2, q)$  as

$$\begin{aligned} \widehat{\nu}^*(b_1, b_2, q) &= \widehat{M}^*(b_2, q) \widehat{W}^*(b_1, q) - \widehat{M}^*(b_1, q) \widehat{W}^*(b_2, q), \text{ where} \\ \widehat{M}^*(b, q) &= \frac{1}{S} \sum_{i,\ell} \left( B_{i,\ell}^* \mathbf{1}\left(b \leq B_{i,\ell}^* \leq b + \frac{a}{q}\right) \right. \\ &\quad \left. + \frac{1}{N-1} \left( \mathbf{1}\left(B_{i,\ell}^* \leq b + \frac{a}{q}\right) \left(b + \frac{a}{q} - B_{i,\ell}^*\right) - \mathbf{1}(B_{i,\ell}^* \leq b)(b - B_{i,\ell}^*) \right) \right), \\ \widehat{W}^*(b, q) &= \frac{1}{S} \sum_{i,\ell} \mathbf{1}\left(b \leq B_{i,\ell}^* \leq b + \frac{a}{q}\right). \end{aligned} \quad (\text{A.4})$$

The bootstrapped process is defined as  $\Phi^*(\cdot, \cdot, \cdot) = \sqrt{S}(\widehat{\nu}^*(\cdot, \cdot, \cdot) - \widehat{\nu}(\cdot, \cdot, \cdot))$ .<sup>18</sup> Define the GMS

<sup>18</sup>Note that one can also use the bootstrap estimator for  $\sigma_\nu^2(b_1, b_2, q)$  and we will introduce this in Section 4.2.

function  $\psi_S(b_1, b_2, q)$  as

$$\psi_S(b_1, b_2, q) = -\beta_S \cdot 1\left(\frac{\sqrt{S}\widehat{\nu}(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} < -\kappa_S\right). \quad (\text{A.5})$$

For a significance level  $\alpha < 1/2$ , define the bootstrapped critical value  $\hat{c}_{\eta, au}$  as

$$\hat{c}_{\eta, au} = \sup \left\{ c \mid P^* \left( \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi^*(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} + \psi_S(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q) \leq c \right) \leq 1 - \alpha + \eta \right\} + \eta.$$

The decision rule is: “Reject  $H_{0, au}$  ( $H'_{0, au}$ ) when  $\widehat{T}_{S, au} > \hat{c}_{\eta, au}$ .”

A rejection of  $H_{0, au}$  ( $H'_{0, au}$ ) would indicate that we have a strong evidence that the key equilibrium assumption, monotonicity of the equilibrium bids, does not hold in the data and the structural model is invalid. The following theorem shows the asymptotic size control and the power against fixed alternatives of our test.

**Theorem A.1** *Suppose Assumptions 2.9 and 3.1 hold. Then,*

- (a) *under  $H_0$ ,  $\lim_{S \rightarrow \infty} P(\widehat{T}_{S, au} > \hat{c}_{\eta, au}) \leq \alpha$ ;*
- (b) *under  $H_1$ ,  $\lim_{S \rightarrow \infty} P(\widehat{T}_{S, au} > \hat{c}_{\eta, au}) = 1$ .*

Theorem A.1 shows that our test has asymptotic size control under the null hypothesis and is consistent against fixed alternatives. We briefly summarize the implementation procedure of the test as follows.

### Implementation Procedure for the Test under Homogeneous Auctions:

1. Estimate  $\nu(b_1, b_2, q)$  by  $\widehat{\nu}(b_1, b_2, q) = \widehat{M}(b_2, q)\widehat{W}(b_1, q) - \widehat{M}(b_1, q)\widehat{W}(b_2, q)$  according to (A.1).
2. Compute  $\widehat{\sigma}_{\nu, \epsilon}^2(b_1, b_2, q)$  according to (A.2).
3. Compute the test statistic  $\widehat{T}_{S, au}$  according to (A.3).
4. Compute the GMS function  $\psi_S(b_1, b_2, q)$  according to (A.5).
5. For  $k = 1, \dots, K$  bootstrap sample, compute  $\widehat{\nu}^{*, k}(b_1, b_2, q)$  and define  $\Phi_t^{*, k}(\cdot) = \sqrt{S}(\widehat{\nu}^{*, k}(b_1, b_2, q) - \widehat{\nu}(b_1, b_2, q))$ .

6. For a significance level  $\alpha < 1/2$ , compute the bootstrapped critical value  $\hat{c}_{\eta, au}$  as the  $(1 - \alpha + \eta)$ -th quantile of

$$\left\{ \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi^{*,k}(b_1, b_2, q)}{\hat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} + \psi_S(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q) \right\} : k = 1, \dots, K$$

plus  $\eta$ .

7. Reject the null if  $\hat{T}_{S, au} > \hat{c}_{\eta, au}$ .

**Proof of Theorem A.1:** Given that

$$\left\{ \left( B_{i, \ell} 1\left(b \leq B_{i, \ell} \leq b + \frac{a}{q}\right) + \frac{1}{N-1} \left( 1\left(B_{i, \ell} \leq b + \frac{a}{q}\right) \left(b + \frac{a}{q} - B_{i, \ell}\right) - 1\left(B_{i, \ell} \leq b\right) (b - B_{i, \ell}) \right) \right) : b \in [\underline{b}, \bar{b}], q = 2, \dots \right\},$$

$$\left\{ 1\left(b \leq B_{i, \ell} \leq b + \frac{a}{q}\right) : b \in [\underline{b}, \bar{b}], q = 2, \dots \right\},$$

are both Vapnik-Chervonenkis (VC) classes of functions, we have  $\sqrt{S}(\widehat{M}(\cdot, \cdot) - M(\cdot, \cdot))$  and  $\sqrt{S}(\widehat{W}(\cdot, \cdot) - W(\cdot, \cdot))$  weakly converge to Gaussian processes in the limit with influence functions  $\phi_{i, \ell, M}(\cdot, \cdot)$  and  $\phi_{i, \ell, W}(\cdot, \cdot)$  defined in Section 3.1. Then by a similar argument of Lemma A.2 of Hsu et al. (2019), we have  $\sqrt{S}(\widehat{\nu}(\cdot, \cdot, \cdot) - \nu(\cdot, \cdot, \cdot))$  weakly converging to a Gaussian process in the limit with influence functions  $\phi_{i, \cdot, \nu}(\cdot, \cdot, \cdot)$ . By the same arguments for Lemma A.2 of Hsu et al. (2019), it is true that  $|\widehat{\sigma}_{\nu}(\cdot, \cdot, \cdot) - \sigma_{\nu}(\cdot, \cdot, \cdot)| \xrightarrow{p} 0$  uniformly. Define

$$\sigma_{\nu, \epsilon}^2(b_1, b_2, q) = \max\{\sigma_{\nu}^2(b_1, b_2, q), \epsilon \cdot \sigma_{\nu}^2(\underline{b}, (\underline{b} + \bar{b})/2, 2)\}.$$

It is true that  $|\widehat{\sigma}_{\nu, \epsilon}^{-1}(\cdot, \cdot, \cdot) - \sigma_{\nu, \epsilon}^{-1}(\cdot, \cdot, \cdot)| \xrightarrow{p} 0$  uniformly. Also, the bootstrapped process  $\Phi^*(\cdot, \cdot, \cdot) = \sqrt{S}(\widehat{\nu}^*(\cdot, \cdot, \cdot) - \widehat{\nu}(\cdot, \cdot, \cdot))$  will weakly converges to the same limiting Gaussian process as  $\sqrt{S}(\widehat{\nu}(\cdot, \cdot, \cdot) - \nu(\cdot, \cdot, \cdot))$  conditional on the sample path with probability approaching 1. Given all these results, Theorem A.1 can be proved by similar arguments for Theorem 4.1 and Theorem 5.1 of Hsu et al. (2019) and we omit the details.  $\square$

## B Testing with Heterogeneous Number of Agents

In this appendix, we discuss how to conduct tests when we allow for heterogeneous numbers of agents in each game. For illustration and for simplicity, we focus on the auction case and it is straightforward to extend to other cases. Assume that there are  $T$  different types of auctions with different numbers of bidders. Let  $N_t \geq 2$  denote the number of bidders for type  $t$  auctions for  $t = 1, \dots, T$  and  $T$  is finite. Let  $\xi_t(b)$  be the inverse of the BNE bidding strategy for type  $t$  auctions, and  $g_t(b)$  and  $G_t(b)$  be the PDF and CDF of the bids in type  $t$  auctions. Note that the inverse of the BNE bidding strategy for type  $t$  auctions can be written as

$$\xi_t(b) = b + \frac{1}{N_t - 1} \frac{G_t(b)}{g_t(b)}. \quad (\text{B.1})$$

The null hypothesis in this case is given as:

$$H_{0,T}^t : \xi_t(b) = b + \frac{1}{N_t - 1} \frac{G_t(b)}{g_t(b)} \text{ is weakly increasing in } b \text{ for all } t = 1, \dots, T. \quad (\text{B.2})$$

For each  $t$ , let  $L_t$  denote the number of type  $t$  auctions. Let  $S_t = N_t L_t$  be the total number of bidders for type  $t$  auctions and  $S = \sum_{t=1}^T S_t$  be the total number of bidders in the data.

**Assumption B.1** *Assume that  $S_t/S \rightarrow \delta_t > 0$  for  $t = 1, \dots, T$ .*

Assumption B.1 implies that the number of auctions of each type diverges to infinity at the same rate. Similar to Lemma 3.1,  $H_{0,T}$  in (B.2) is equivalent to

$$\begin{aligned} H'_{0,T} : \nu_t(b_1, b_2, q) = M_t(b_2, q)W_t(b_1, q) - M_t(b_1, q)W_t(b_2, q) \leq 0 \\ \text{for all } t = 1, \dots, T \text{ and for all } (b_1, b_2, q) \in \mathcal{L}, \end{aligned} \quad (\text{B.3})$$

where

$$M_t(b, q) = \left( E \left[ B_{i_t, \ell_t} 1 \left( b \leq B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \right] \right)$$

$$\begin{aligned}
& + \frac{1}{N_t - 1} E \left[ 1 \left( B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - B_{i_t, \ell_t} \right) - 1(B_{i_t, \ell_t} \leq b)(b - B_{i_t, \ell_t}) \right], \\
W_t(b, q) & = E \left[ 1 \left( b \leq B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \right].
\end{aligned}$$

We estimate  $\nu_t(b_1, b_2, q)$  by  $\widehat{\nu}_t(b_1, b_2, q) = \widehat{M}_t(b_2, q)\widehat{W}_t(b_1, q) - \widehat{M}_t(b_1, q)\widehat{W}_t(b_2, q)$  in which

$$\begin{aligned}
\widehat{M}_t(b, q) & = \frac{1}{S_t} \sum_{i_t, \ell_t} \left( B_{i_t, \ell_t} 1 \left( b \leq B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \right. \\
& \quad \left. + \frac{1}{N_t - 1} \left( 1 \left( B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - B_{i_t, \ell_t} \right) - 1(B_{i_t, \ell_t} \leq b)(b - B_{i_t, \ell_t}) \right) \right), \\
\widehat{W}_t(b, q) & = \frac{1}{S_t} \sum_{i_t, \ell_t} 1 \left( b \leq B_{i_t, \ell_t} \leq b + \frac{a}{q} \right),
\end{aligned}$$

where  $i_t$  and  $\ell_t$  are the indexes for those bidders in type  $t$  auctions. Let

$$\begin{aligned}
\widehat{\phi}_{i_t, \ell_t, M_t}(b, q) & = B_{i_t, \ell_t} 1 \left( b \leq B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \\
& \quad + \frac{1}{N_t - 1} \left( 1 \left( B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - B_{i_t, \ell_t} \right) - 1(B_{i_t, \ell_t} \leq b)(b - B_{i_t, \ell_t}) \right) - \widehat{M}_t(b, q), \\
\widehat{\phi}_{i_t, \ell_t, W_t}(b, q) & = 1 \left( b \leq B_{i_t, \ell_t} \leq b + \frac{a}{q} \right) - \widehat{W}_t(b, q), \\
\widehat{\phi}_{i_t, \ell_t, \nu_t}(b_1, b_2, q) & = \widehat{W}_t(b_1, q)\widehat{\phi}_{i_t, \ell_t, M_t}(b_2, q) + \widehat{M}_t(b_2, q)\widehat{\phi}_{i_t, \ell_t, W_t}(b_1, q) \\
& \quad - \widehat{W}_t(b_2, q)\widehat{\phi}_{i_t, \ell_t, M_t}(b_1, q) - \widehat{M}_t(b_1, q)\widehat{\phi}_{i_t, \ell_t, W_t}(b_2, q),
\end{aligned}$$

which are the estimated influence functions for estimators  $\widehat{M}_t(b, q)$ ,  $\widehat{W}_t(b, q)$  and  $\widehat{\nu}_t(b_1, b_2, q)$ , respectively. Define  $\sigma_{\nu_t}^2(b_1, b_2, q) = \kappa_t((b_1, b_2, q), (b_1, b_2, q))$  which is the asymptotic variance of  $\sqrt{S_t}(\widehat{\nu}_t(b_1, b_2, q) - \nu_t(b_1, b_2, q))$  and its estimator as  $\widehat{\sigma}_{\nu_t}^2(b_1, b_2, q)$

$$\widehat{\sigma}_{\nu_t}^2(b_1, b_2, q) = \frac{1}{S_t} \sum_{i_t, \ell_t} \widehat{\phi}_{i_t, \ell_t, \nu_t}^2(b_1, b_2, q).$$

For some small  $\epsilon > 0$ , define

$$\widehat{\sigma}_{\nu_t, \epsilon}^2(b_1, b_2, q) = \max\{\widehat{\sigma}_{\nu_t}^2(b_1, b_2, q), \epsilon \cdot \widehat{\sigma}_{\nu_t}^2(\underline{b}, (\underline{b} + \bar{b})/2, 2)\}.$$

Define our test statistic as

$$\widehat{T}_{S,T} = \sum_{t=1}^T \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \sqrt{S_t} \frac{\widehat{\nu}_t(b_1, b_2, q)}{\widehat{\sigma}_{\nu_t, \epsilon}(b_1, b_2, q)}, 0 \right\}^2 Q(b_1, b_2, q).$$

For the nonparametric bootstrap, we bootstrap auctions within each type. To be specific, for each  $t$ , let  $\{(\{B_{1, \ell_t}^*, \dots, B_{N_t \ell_t}^*\}) : \ell_t \leq L_t\}$  be a bootstrap subsample in which  $B_{i_t, \ell_t}^* = B_{i_t \ell_t}^*$  and  $\{\ell_t^* : \ell_t \leq L_t\}$  is an i.i.d. bootstrap sample drawn from the empirical distribution of  $\{\ell_t : \ell_t \leq L_t\}$ . Then we collect all  $T$  bootstrap subsamples together. By doing this, the total number of bidders in each bootstrap sample remains as  $S$ . Define the bootstrap estimator for  $\nu_t(b_1, b_2, q)$  as  $\widehat{\nu}_t^*(b_1, b_2, q) = \widehat{M}_t^*(b_2, q) \widehat{W}_t^*(b_1, q) - \widehat{M}_t^*(b_1, q) \widehat{W}_t^*(b_2, q)$  in which

$$\begin{aligned} \widehat{M}_t^*(b, q) &= \frac{1}{S_t} \sum_{i_t, \ell_t} \left( B_{i_t, \ell_t}^* \mathbf{1} \left( b \leq B_{i_t, \ell_t}^* \leq b + \frac{a}{q} \right) \right. \\ &\quad \left. + \frac{1}{N_t - 1} \left( \mathbf{1} \left( B_{i_t, \ell_t}^* \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - B_{i_t, \ell_t}^* \right) - \mathbf{1} \left( B_{i_t, \ell_t}^* \leq b \right) (b - B_{i_t, \ell_t}^*) \right) \right), \\ \widehat{W}_t^*(b, q) &= \frac{1}{S_t} \sum_{i_t, \ell_t} \mathbf{1} \left( b \leq B_{i_t, \ell_t}^* \leq b + \frac{a}{q} \right). \end{aligned}$$

The bootstrapped process is defined as  $\Phi_t^*(\cdot) = \sqrt{S_t}(\widehat{\nu}_t^*(b_1, b_2, q) - \widehat{\nu}_t(b_1, b_2, q))$ . Define the GMS function  $\psi_S(t, b_1, b_2, q)$  as

$$\psi_{S_t}(t, b_1, b_2, q) = -\beta_{S_t} \cdot \mathbf{1} \left( \frac{\sqrt{S_t} \widehat{\nu}_t(b_1, b_2, q)}{\widehat{\sigma}_{\nu_t, \epsilon}(b_1, b_2, q)} < -\kappa_{S_t} \right).$$

For a significance level  $\alpha < 1/2$ , define the bootstrapped critical value  $\widehat{c}_{\eta, T}$  as

$$\begin{aligned} \widehat{c}_{\eta, T} &= \sup \left\{ c \mid P^* \left( \sum_{t=1}^T \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi_t^*(b_1, b_2, q)}{\widehat{\sigma}_{\nu_t, \epsilon}(b_1, b_2, q)} + \psi_{S_t}(t, b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q) \leq c \right) \right. \\ &\quad \left. \leq 1 - \alpha + \eta \right\} + \eta. \end{aligned}$$

Then we reject  $H'_{0,t}$  when  $\widehat{T}_{S,T} > \widehat{c}_{\eta, T}$ . We summarize the size and power result of the test.

**Theorem B.1** *Suppose that the unobserved the bidders' valuations  $\{V_{i, \ell_t} : i = 1, \dots, N_t, \ell_t = 1, \dots, L_t\}$  satisfy Assumption 3.1 for each  $t = 1, \dots, T$ . Suppose that Assumptions 2.9 and*

B.1 hold. Then,

(a) under  $H_0$ ,  $\lim_{S \rightarrow \infty} P(\widehat{T}_{S,T} > \hat{c}_{\eta,T}) \leq \alpha$ ;

(b) under  $H_1$ ,  $\lim_{S \rightarrow \infty} P(\widehat{T}_{S,T} > \hat{c}_{\eta,T}) = 1$ .

Note that, for the test with different numbers of bidders, technically we are implementing a joint test for  $T$  types of auctions.

**Remark B.1** *In the main text, we also consider nonparametric test with covariates and semiparametric test. For these two cases, we can extend these tests to allow for heterogeneous numbers of bidders in a similar way. Therefore, we omit the details.*

## C A Semiparametric Test for Auction Games

In the auction model, the main idea of this semiparametric approach via homogenization of bids is to impose a parametric specification on  $\Gamma(x; \theta)$  up to a scalar. Then once we parametrically estimate  $\hat{\theta}$ , we can use the estimated factor  $\Gamma(x; \hat{\theta})$  as the rescaling factor. Then we can apply the test developed in Section 3.1 with  $\widehat{B}_{i,\ell}^u = B_{i,\ell} / \Gamma(X_\ell; \hat{\theta})$  after we account for the estimation effect of  $\hat{\theta}$ .

Following Haile et al. (2003) and Ma et al. (2021), we assume that  $\Gamma(X_\ell; \theta) = \exp(X_\ell \theta)$  and that

$$\log(B_{i,\ell}) = X_\ell \theta + \log(B_{i,\ell}^u), \text{ with } E[\log(B_{i,\ell}^u)] = 0, \quad (\text{C.1})$$

where, the first element of  $X_\ell$  is a constant term. Then we can estimate  $\theta$  consistently by an ordinary least squares (OLS) estimator. We get  $\{\widehat{B}_{i,\ell}^u = \exp(-X_\ell \hat{\theta}) B_{i,\ell} : i = 1, \dots, N, \ell = 1, \dots, L\}$ . Then we can treat  $\widehat{B}_{i,\ell}^u$  as the observed bids and construct the test statistic as in Section 3.1. However, to construct the critical value, we have to account for the estimation effect of  $\hat{\theta}$  and the estimation effect makes the influence function representations for the estimated moments too complicated. Hence, to avoid estimating the asymptotic standard errors of the moments from the estimated influence functions as in previous cases, we suggest to use the bootstrap standard error estimators instead. We briefly summarize the implementation procedure of the test as follows.

**Implementation Procedure for Semiparametric Test under Observed Auction Heterogeneity:**

1. Estimate  $\theta$  by  $\hat{\theta}$  which is given as

$$\hat{\theta} = \left( \frac{1}{S} \sum_{i,\ell} X'_\ell X_\ell \right)^{-1} \left( \frac{1}{S} \sum_{i,\ell} X'_\ell \log(B_{i,\ell}) \right),$$

and get  $\hat{B}_{i,\ell}^u = \exp(-X_\ell \hat{\theta}) B_{i,\ell}$ .

2. Estimate  $\nu(b_1, b_2, q, \theta)$  by  $\hat{\nu}(b_1, b_2, q, \hat{\theta}) = \widehat{M}(b_2, q, \hat{\theta}) \widehat{W}(b_1, q, \hat{\theta}) - \widehat{M}(b_1, q, \hat{\theta}) \widehat{W}(b_2, q, \hat{\theta})$  in which

$$\begin{aligned} \widehat{M}(b, q, \hat{\theta}) &= \frac{1}{S} \sum_{i,\ell} \left( \hat{B}_{i,\ell}^u 1\left(b \leq \hat{B}_{i,\ell}^u \leq b + \frac{a}{q}\right) \right. \\ &\quad \left. + \frac{1}{N-1} \left( 1\left(\hat{B}_{i,\ell}^u \leq b + \frac{a}{q}\right) \left(b + \frac{a}{q} - \hat{B}_{i,\ell}^u\right) - 1\left(\hat{B}_{i,\ell}^u \leq b\right) (b - \hat{B}_{i,\ell}^u) \right) \right), \\ \widehat{W}(b, q, \hat{\theta}) &= \frac{1}{S} \sum_{i,\ell} 1\left(b \leq \hat{B}_{i,\ell}^u \leq b + \frac{a}{q}\right). \end{aligned}$$

3. For  $k = 1, \dots, K$  bootstrap sample, compute  $\hat{\theta}^{*,k}$ , the bootstrap estimator for  $\theta$ , and obtain  $\hat{u}_{i,\ell}^{*,k} = \exp(-X_{i,\ell}^{*,k} \hat{\theta}^{*,k}) B_{i,\ell}^{*,k}$  and  $\hat{\nu}^{*,k}(b_1, b_2, q, \hat{\theta}^{*,k})$ .
4. Compute the bootstrap standard error estimate for  $\sigma_\nu(b_1, b_2, q, \hat{\theta})$  as

$$\begin{aligned} \hat{\sigma}_\nu^*(b_1, b_2, q, \hat{\theta}) &= \left( \frac{1}{K} \sum_{k=1}^K S(\hat{\nu}^{*,k}(b_1, b_2, q, \hat{\theta}^{*,k}) - \bar{\nu}^*(b_1, b_2, q, \hat{\theta}))^2 \right)^{1/2}, \text{ where} \\ \bar{\nu}^*(b_1, b_2, q, \hat{\theta}) &= \frac{1}{K} \sum_{k=1}^K \hat{\nu}^{*,k}(b_1, b_2, q, \hat{\theta}^{*,k}), \end{aligned}$$

and define  $\hat{\sigma}_{\nu,\epsilon}^*(b_1, b_2, q, \hat{\theta}) = \max\{\hat{\sigma}_\nu^*(b_1, b_2, q, \hat{\theta}), \sqrt{\epsilon} \cdot \hat{\sigma}_\nu^*(\underline{b}, (\underline{b} + \bar{b})/2, 2, \hat{\theta})\}$ .

5. Compute the test statistic as

$$\hat{T}_{S,se} = \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \sqrt{S} \frac{\hat{\nu}(b_1, b_2, q, \hat{\theta})}{\hat{\sigma}_{\nu,\epsilon}^*(b_1, b_2, q, \hat{\theta})}, 0 \right\}^2 Q(b_1, b_2, q).$$

6. Compute the GMS function  $\psi_S(t, b_1, b_2, q)$  as

$$\psi_S(t, b_1, b_2, q) = -\beta_S \cdot 1\left(\frac{\sqrt{S}\widehat{\nu}(b_1, b_2, q, \widehat{\theta})}{\widehat{\sigma}_{\nu, \epsilon}^*(b_1, b_2, q, \widehat{\theta})} < -\kappa_S\right).$$

7. Let  $\Phi^{*,k}(b_1, b_2, q) = \sqrt{S}(\widehat{\nu}^{*,k}(b_1, b_2, q, \widehat{\theta}^{*,k}) - \widehat{\nu}(b_1, b_2, q, \widehat{\theta}))$ .

8. For a significance level  $\alpha < 1/2$ , compute  $\widehat{c}$  as the  $(1 - \alpha + \eta)$ -th quantile of

$$\left\{ \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi^{*,k}(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}^*(b_1, b_2, q, \widehat{\theta})} + \psi_S(t, b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q) : k = 1, \dots, K \right\}$$

and define the bootstrapped critical value  $\widehat{c}_{\eta, se} = \widehat{c} + \eta$ .

9. Reject the null if  $\widehat{T}_{S, se} > \widehat{c}_{\eta, se}$ .

We summarize the asymptotic properties of the semiparametric test in the Appendix. Note that the bootstrap estimator for the asymptotic standard error in Step 5 can be applied to other cases as well. However, it is more time-consuming to compute bootstrap standard error estimator than to compute the standard error estimator from the estimated influence functions. Also, it is straightforward to allow for heterogeneous number of bidders as in Section B and we omit the details.

## D Asymptotics of Semiparametric Test in Section 4.2

This section provides additional assumptions and asymptotic results of the semiparametric test developed in Section 4.2.

We impose the assumption on the first stage estimator,  $\widehat{\theta}$ .

**Assumption D.1** *Assume that (i)  $\widehat{\theta}$  is an M-estimator of  $\theta$  so that*

$$\widehat{\theta} \equiv \arg \max_{\theta \in \Theta} \frac{1}{S} \sum_{i, \ell} \zeta(B_{i, \ell}, X_{\ell}, \theta),$$

for some  $\zeta$  function and  $\{\zeta(B_{i, \ell}, X_{\ell}, \theta) : \theta \in \Theta\}$  is a Vapnik-Chervonenkis (VC) class of functions.

(ii) There is a point  $\theta_0$  in the interior of the compact parameter space  $\Theta$  so that

$$\sqrt{n}(\widehat{\theta} - \theta_0) = \frac{1}{\sqrt{S}} \sum_{i,\ell} \psi_\theta(B_{i,\ell}, X_\ell, \theta_0) + o_p(1),$$

where  $\psi_\theta$  is a given function of  $B_{i,\ell}$  and  $X_\ell$  with  $E[\psi_\theta(Y, X, \theta_0)] = 0$  and  $E\|\psi_\theta(Y, X, \theta_0)\|^{2+\epsilon} < \infty$  for some  $\epsilon > 0$ .

(iii) The first stage bootstrap estimator  $\widehat{\theta}^*$  satisfies that

$$\sqrt{n}(\widehat{\theta}^* - \widehat{\theta}) = \frac{1}{\sqrt{S}} \sum_{i,\ell} (P_i - 1) \cdot \psi_\theta(B_{i,\ell}, X_\ell, \theta_0) + o_p(1),$$

$\{P_i : i = 1, \dots, N\}$  are i.i.d. Poisson random variables with mean 1 and are independent of the sample path.

Assumption D.1 states that  $\widehat{\theta}$  is an  $M$ -estimator with an asymptotically linear representation, implying that  $\widehat{\theta}$  is asymptotically normally distributed.

**Assumption D.2** Assume that  $0 < \delta \leq \Gamma(x, \theta) \leq M < \infty$  for all  $x \in \mathcal{X}$  and  $\theta \in \Theta$ .

Assumption D.2 implies that  $1/\Gamma(x, \theta)$  is uniformly bounded above and uniformly bounded away from zero.

**Assumption D.3** Assume that for all  $x \in \mathcal{X}$ ,  $\Gamma(x, \theta)$  is continuously differentiable in  $\theta$  at with bounded derivatives.

Assumption D.3 implies that  $\{1(b \leq B_{i,\ell}/\Gamma(X_\ell, \theta) \leq b + r) : \theta \in \Theta, b \in [b, \bar{b}], r \in [0, a]\}$  is a VC class of functions which is crucial for our empirical process results. Let

$$\begin{aligned} M(b, q, \theta) &= E \left[ \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} 1 \left( b \leq \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} \leq b + \frac{a}{q} \right) \right] \\ &\quad + \frac{1}{N-1} E \left[ 1 \left( \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} \leq b + \frac{a}{q} \right) \left( b + \frac{a}{q} - \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} \right) - 1 \left( \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} \leq b \right) \left( b - \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} \right) \right], \\ W(b, q, \theta) &= E \left[ 1 \left( b \leq \frac{B_{i,\ell}}{\Gamma(X_\ell, \theta)} \leq b + \frac{a}{q} \right) \right], \end{aligned}$$

where  $M(b, q, \theta)$  and  $W(b, q, \theta)$  are similar to those in (3.5) and (3.6) except that in  $M(b, q, \theta)$  and  $W(b, q, \theta)$ , we have rescaled bids instead of the original ones.

**Assumption D.4** Assume that for all  $b \in [b, \bar{b}]$ , and for all  $q$ , (i)  $\nabla_{\theta}M(b, q, \theta)$  and  $\nabla_{\theta}M(b, q, \theta)$  are continuous in  $\theta$  for all  $\theta$  with  $\|\theta - \theta_0\| \leq \epsilon$  for some small positive  $\epsilon$ ; (ii)  $\|\nabla_{\theta}M(b, q, \theta)\| \leq M$  and  $\|\nabla_{\theta}M(b, q, \theta)\| \leq M$  for some  $M < \infty$ .

With a bit abuse of notation, we define

$$\begin{aligned} M_{i,\ell}(b, q, \theta) &= \frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)} 1\left(b \leq \frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)} \leq b + \frac{a}{q}\right) \\ &\quad + \frac{1}{N-1} 1\left(\frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)} \leq b + \frac{a}{q}\right) \left(b + \frac{a}{q} - \frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)}\right) - 1\left(\frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)} \leq b\right) \left(b - \frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)}\right), \\ W_{i,\ell}(b, q, \theta) &= 1\left(b \leq \frac{B_{i,\ell}}{\Gamma(X_{\ell}, \theta)} \leq b + \frac{a}{q}\right), \end{aligned}$$

Let  $\widehat{M}(b, q, \theta) = S^{-1} \sum_{i,\ell} M_{i,\ell}(b, q, \theta)$  and  $\widehat{W}(b, q, \theta) = S^{-1} \sum_{i,\ell} W_{i,\ell}(b, q, \theta)$ . Define  $\widehat{v}(b_1, b_2, q, \widehat{\theta}) = \widehat{M}(b_2, q, \widehat{\theta})\widehat{W}(b_1, q, \widehat{\theta}) - \widehat{M}(b_1, q, \widehat{\theta})\widehat{W}(b_2, q, \widehat{\theta})$

**Lemma D.1** Suppose Assumptions 4.2, D.1, D.2, D.3 and D.4 hold. Then it is true that

$$\begin{aligned} &\sqrt{S}(\widehat{M}(b, q, \widehat{\theta}) - M(b, q, \theta_0)) \\ &= \frac{1}{\sqrt{S}} \sum_{i,\ell} M_{i,\ell}(b, q, \theta_0) - M(b, q, \theta_0) + \nabla_{\theta}M(b, q, \theta_0)\psi_{\theta}(B_{i,\ell}, X_{\ell}, \theta_0) + o_p(1) \end{aligned} \quad (\text{D.1})$$

$$\begin{aligned} &\equiv \frac{1}{\sqrt{S}} \sum_{i,\ell} \phi_{i,\ell,M}(b, q, \theta_0) + o_p(1) \\ &\sqrt{S}(\widehat{W}(b, q, \widehat{\theta}) - W(b, q, \theta_0)) \\ &= \frac{1}{\sqrt{S}} \sum_{i,\ell} W_{i,\ell}(b, q, \theta_0) - W(b, q, \theta_0) + \nabla_{\theta}W(b, q, \theta_0)\psi_{\theta}(B_{i,\ell}, X_{\ell}, \theta_0) + o_p(1) \end{aligned} \quad (\text{D.2})$$

$$\equiv \frac{1}{\sqrt{S}} \sum_{i,\ell} \phi_{i,\ell,W}(b, q, \theta_0) + o_p(1)$$

uniformly over  $b \in [b, \bar{b}]$  and  $q$ .

Lemma D.1 provides the influence function representations of  $\widehat{M}(b, q, \widehat{\theta})$  and  $\widehat{W}(b, q, \widehat{\theta})$ . It implies that both  $\sqrt{S}(\widehat{M}(b, q, \widehat{\theta}) - M(b, q, \theta_0))$  and  $\sqrt{S}(\widehat{W}(b, q, \widehat{\theta}) - W(b, q, \theta_0))$  weakly converge to Gaussian processes.

**Lemma D.2** *Suppose Assumptions 4.2, D.1, D.2, D.3 and D.4 hold. Then it is true that*

$$\sqrt{S}(\widehat{\nu}(b_1, b_2, q, \widehat{\theta}) - \nu(b_1, b_2, q, \theta_0)) \equiv \frac{1}{\sqrt{S}} \sum_{i,\ell} \phi_{i,\ell,\nu}(b_1, b_2, q, \theta_0) + o_p(1), \quad \text{where} \quad (\text{D.3})$$

$$\begin{aligned} \phi_{i,\ell,\nu}(b_1, b_2, q, \theta_0) &= W(b_1, q, \theta_0) \phi_{i,\ell,M}(b_2, q, \theta_0) + M(b_2, q, \theta_0) \phi_{i,\ell,W}(b_1, q, \theta_0) \\ &\quad - W(b_2, q, \theta_0) \phi_{i,\ell,M}(b_1, q, \theta_0) - M(b_1, q, \theta_0) \phi_{i,\ell,W}(b_2, q, \theta_0), \end{aligned}$$

uniformly over  $(b_1, b_2, q) \in \mathcal{L}$ .

Lemma D.2 provides the influence function representation of  $\widehat{\nu}(b_1, b_2, q, \widehat{\theta})$ . It follows that  $\sqrt{S}(\widehat{\nu}(b_1, b_2, q, \widehat{\theta}) - \nu(b_1, b_2, q, \theta_0))$  weakly converges to a Gaussian process.

Let  $\widehat{\theta}^*$  and  $\widehat{\nu}^*(b_1, b_2, q, \widehat{\theta}^*)$  denote the bootstrap estimators for  $\theta_0$  and  $\nu(b_1, b_2, q, \theta_0)$ . Let  $\sigma_\nu^2(b_1, b_2, q, \theta_0)$  denote the asymptotic variance of  $\sqrt{S}(\widehat{\nu}(b_1, b_2, q, \widehat{\theta}) - \nu(b_1, b_2, q, \widehat{\theta}))$ . Also define,  $\sigma_{\nu,\epsilon}(b_1, b_2, q, \theta_0) = \max\{\sigma_\nu(b_1, b_2, q), \sqrt{\epsilon} \cdot \sigma_\nu(\underline{b}, (\underline{b} + \bar{b})/2, 2, \theta_0)\}$ . Let the bootstrap standard error estimator for  $\sigma_\nu(b_1, b_2, q, \theta_0)$  as

$$\begin{aligned} \widehat{\sigma}_\nu^*(b_1, b_2, q, \widehat{\theta}) &= \left( \frac{1}{K} \sum_{k=1}^K S(\widehat{\nu}^{*,k}(b_1, b_2, q, \widehat{\theta}^{*,k}) - \bar{\nu}^*(b_1, b_2, q, \widehat{\theta}))^2 \right)^{1/2}, \quad \text{where} \\ \bar{\nu}^*(b_1, b_2, q, \widehat{\theta}) &= \frac{1}{K} \sum_{k=1}^K \widehat{\nu}^{*,k}(b_1, b_2, q, \widehat{\theta}^{*,k}), \end{aligned}$$

and define  $\widehat{\sigma}_{\nu,\epsilon}^*(b_1, b_2, q, \widehat{\theta}) = \max\{\widehat{\sigma}_\nu^*(b_1, b_2, q), \sqrt{\epsilon} \cdot \widehat{\sigma}_\nu^*(\underline{b}, (\underline{b} + \bar{b})/2, 2, \widehat{\theta})\}$ .

**Lemma D.3** *Suppose Assumptions 4.2, D.1, D.2, D.3 and D.4 hold. Then it is true that uniformly over  $(b_1, b_2, q) \in \mathcal{L}$*

$$\begin{aligned} &\sqrt{S}(\widehat{\nu}^*(b_1, b_2, q, \widehat{\theta}^*) - \nu(b_1, b_2, q, \widehat{\theta})) \\ &\equiv \frac{1}{\sqrt{S}} \sum_{i,\ell} (P_i - 1) \cdot \phi_{i,\ell,\nu}(b_1, b_2, q, \theta_0) + o_p(1), \quad \text{where} \end{aligned} \quad (\text{D.4})$$

$\{P_i : i = 1, \dots, N\}$  are the same as those in Assumption D.1(iii). In addition, uniformly over  $(b_1, b_2, q) \in \mathcal{L}$ ,

$$\lim_{K \rightarrow \infty} |\widehat{\sigma}_{\nu,\epsilon}^*(b_1, b_2, q, \widehat{\theta}) - \sigma_{\nu,\epsilon}(b_1, b_2, q, \theta_0)| \xrightarrow{P} 0 \quad (\text{D.5})$$

Lemma D.3 can be proved by a similar procedure in Theorem 3.6.1 of van der Vaart and Wellner (1996). Lemma D.3 implies the bootstrap validity of our bootstrap estimators in that it can approximate the limiting process of  $\sqrt{S}(\widehat{\nu}(b_1, b_2, q, \widehat{\theta}) - \nu(b_1, b_2, q, \theta_0))$  well. The second part of it shows the uniform consistency of the bootstrap standard error estimators.

**Theorem D.1** *Suppose Assumptions 4.2, D.1, D.2, D.3 and D.4 hold. Then the following results regarding the semiparametric test proposed in Section 4.2 in which the test statistic,  $\widehat{T}_{S,se}$ , and critical value  $\widehat{c}_{\eta,se}$  are generated as in “Implementation Procedure for Semiparametric Test under Observed Auction Heterogeneity”. Then*

- (a) *under the null hypothesis,  $\lim_{S \rightarrow \infty} P(\widehat{T}_{S,se} > \widehat{c}_{\eta,se}) \leq \alpha$ ;*
- (b) *under the alternative hypothesis,  $\lim_{S \rightarrow \infty} P(\widehat{T}_{S,se} > \widehat{c}_{\eta,se}) = 1$ .*

The proof for Theorem D.1 is similar to that for Theorem 2.2 given the lemmas in this section and we omit the details.

## E Semiparametric Test under Unobserved Game Heterogeneity

We extend the semiparametric test in Section 4.2 to allow for unobserved heterogeneity. In addition to the setup in Section 4.2, we assume there exists an unobserved vector of  $Z$  denoting the unobserved characteristics. As in Section 4.2, we impose a semiparametric model to homogenize the bids. Let  $\Gamma(x)$  be a strictly positive function of covariates. We assume that Assumption 4.3 holds.

**Assumption E.1** *Assume that (i)  $\{(V_{1,\ell} = \Gamma(X_\ell)\Lambda(Z_\ell, u_{1,\ell}), V_{2,\ell} = \Gamma(X_\ell)\Lambda(Z_\ell, u_{2,\ell}), \dots, V_{N,\ell} = \Gamma(X_\ell)\Lambda(Z_\ell, u_{N,\ell}), X_\ell, Z_\ell) : \ell = 1, \dots, L\}$  are i.i.d. random vectors; (ii) for each  $\ell$ , conditional on  $Z_\ell = z$ ,  $\{\Lambda(Z_\ell, u_{i,\ell}) : i = 1, \dots, N\}$  are i.i.d. random variables with conditional PDF  $f_\lambda(\cdot | Z = z)$  and conditional CDF  $F_\lambda(\cdot | Z = z)$  and are independent of  $X_\ell$ ; (iii)  $f_\lambda(\cdot | Z = z)$  is strictly positive and bounded away from zero on its support, a compact interval  $[\underline{\lambda}_z, \bar{\lambda}_z] \subseteq R$ , and is twice continuously differentiable on  $(\underline{\lambda}_z, \bar{\lambda}_z)$ .*

The condition that for each  $\ell$ , conditional on  $Z_\ell = z$ ,  $\{\Lambda(Z_\ell, u_{i,\ell}) : \ell = 1, \dots, L\}$  are i.i.d.

random variables is implied by that for each  $\ell$ , conditional on  $Z_\ell = z$ ,  $\{u_{i,\ell} : \ell = 1, \dots, L\}$  are i.i.d. random variables. We assume that  $V_{i,\ell} = \Gamma(X_\ell)\Lambda(Z_\ell, u_{i,\ell})$  so that we can rescale the actions by a function of observables.

Then the same argument in Section 4.2 implies that

$$\begin{aligned} \xi(b, x, z) &\equiv \frac{T'(b, x, z)}{P'(b, x, z)} \text{ is strictly increasing in } b \text{ for all } x \in \mathcal{X} \text{ and for all } z \in \mathcal{Z}. \\ \Leftrightarrow \xi(b^\lambda, z) &\equiv \frac{T^{\lambda'}(b^\lambda, z)}{P^{\lambda'}(b^\lambda, z)} \text{ is strictly increasing in } b^\lambda \text{ for all } z \in \mathcal{Z}. \end{aligned}$$

Combining Section 4.2 and Section 4.3, we can formulate the testable implications as

$$H'_{0,\text{un,semi}} : \nu(b_1^\lambda, b_2^\lambda, q) = M(b_2^\lambda, q)W(b_1^\lambda, q) - M(b_1^\lambda, q)W(b_2^\lambda, q) \leq 0 \text{ for all } (b_1^\lambda, b_2^\lambda, q) \in \mathcal{L},$$

where

$$\begin{aligned} M(b^\lambda, q) &= \int_0^1 \int_{b^\lambda}^{b^\lambda + \frac{a}{q}} h(\tilde{b}, z) T'(\tilde{b}, z) d\tilde{b} dz, \quad W(b^\lambda, q) = \int_0^1 \int_{b^\lambda}^{b^\lambda + \frac{a}{q}} h(\tilde{b}, z) P'(\tilde{b}, z) d\tilde{b} dz, \text{ and} \\ \mathcal{L} &= \left\{ (b_1^\lambda, b_2^\lambda, q) : \left( \frac{b_1^\lambda - \underline{b}^\lambda}{a}, \frac{b_2^\lambda - \underline{b}^\lambda}{a} \right) \cdot q \in (0, 1, \dots, q)^2, b_1^\lambda > b_2^\lambda \text{ for } q=2,3,\dots \right\}. \end{aligned}$$

Then we can apply the test in Section 4.2 and we omit the details.

## F Proofs for Theorems

**Proof of Theorem 2.1:** Consider a symmetric Bayesian game with scalar private type  $v \in [\underline{v}, \bar{v}]$  and scalar action  $b \in [\underline{b}, \bar{b}]$ . Given opponents' action vector  $B_{-i}$ , agent  $i$ 's interim payoff from choosing  $b$  is

$$\pi(b, v; B_{-i}) = v D(b, B_{-i}) - C(b, B_{-i}). \quad (\text{F.1})$$

Fix a continuously differentiable CDF  $G$  on  $[\underline{b}, \bar{b}]$  with density  $g > 0$  on  $(\underline{b}, \bar{b})$ . Define the

reduced-form (expected) payoff

$$U(b, v; G) := vP(b; G) - T(b; G), \quad P(b; G) := E_{B_{-i}}[D(b, B_{-i})], \quad T(b; G) := E_{B_{-i}}[C(b, B_{-i})], \quad (\text{F.2})$$

where the expectation is taken under  $B_{-i} \sim G^{\otimes(N-1)}$  and the expectations exist.

Assume  $P(\cdot; G)$  and  $T(\cdot; G)$  extend continuously to  $[\underline{b}, \bar{b}]$  and satisfy  $P(\cdot; G), T(\cdot; G) \in C^2(\underline{b}, \bar{b})$  with  $P'(b; G) > 0$  for all  $b \in (\underline{b}, \bar{b})$ . Define the inverse-type map

$$\xi(b; G) := \frac{T'(b; G)}{P'(b; G)}, \quad b \in (\underline{b}, \bar{b}), \quad (\text{F.3})$$

and assume  $\xi(\cdot; G)$  extends continuously to  $[\underline{b}, \bar{b}]$ , with  $\underline{\xi} := \xi(\underline{b}; G)$  and  $\bar{\xi} := \xi(\bar{b}; G)$ .

**Necessity.** Suppose  $G$  is rationalizable by a strictly monotone interior symmetric equilibrium, in the sense that there exist a type distribution  $F$  on  $[\underline{v}, \bar{v}]$  and a symmetric equilibrium strategy  $s : [\underline{v}, \bar{v}] \rightarrow [\underline{b}, \bar{b}]$  such that:

- (i)  $s$  is continuous and strictly increasing;
- (ii) (C1) actions  $(B_1, \dots, B_N)$  are i.i.d. with common CDF  $G$ .
- (iii) for every  $v \in (\underline{v}, \bar{v})$ , the action  $b = s(v)$  is an interior maximizer of  $U(b, v; G)$  on  $[\underline{b}, \bar{b}]$ .

Then the inverse-type map is strictly increasing on  $(\underline{b}, \bar{b})$  and satisfies

$$\xi(b; G) = s^{-1}(b) \quad \text{for all } b \in (\underline{b}, \bar{b}). \quad (\text{F.4})$$

**Proof for Necessity:** Suppose there exists  $(F, s)$  satisfying (i)–(iii) in the theorem statement. By (ii), in the symmetric equilibrium the opponents' action profile  $B_{-i}$  is i.i.d. with CDF  $G$ , so a type- $v$  agent evaluates expected payoff  $U(b, v; G) = vP(b; G) - T(b; G)$ . By (iii),  $b = s(v) \in (\underline{b}, \bar{b})$  is an interior maximizer for each  $v \in (\underline{v}, \bar{v})$ , so the first-order condition holds:

$$0 = \frac{\partial}{\partial b} U(b, v; G) \Big|_{b=s(v)} = vP'(s(v); G) - T'(s(v); G).$$

Since  $P'(b; G) > 0$  on  $(\underline{b}, \bar{b})$ , dividing yields

$$v = \frac{T'(s(v); G)}{P'(s(v); G)} = \xi(s(v); G), \quad v \in (\underline{v}, \bar{v}).$$

Because  $s$  is strictly increasing and maps  $(\underline{v}, \bar{v})$  onto  $(\underline{b}, \bar{b})$ , for each  $b \in (\underline{b}, \bar{b})$  there exists a unique  $v = \xi(b; G) = s^{-1}(b)$  such that  $b = s(v)$ . Since  $s^{-1}$  is strictly increasing,  $\xi(\cdot; G)$  is strictly increasing on  $(\underline{b}, \bar{b})$ .

We now show that the rationalizing type distribution  $F$  is unique. Since  $\xi(b; G) = s^{-1}(b)$  on  $(\underline{b}, \bar{b})$ , the equilibrium type satisfies  $V = s^{-1}(B) = \xi(B; G)$ . Under (C1),  $B \sim G$ , and  $\xi(\cdot; G)$  is uniquely determined by  $G$ . Therefore, the rationalizing  $F$  is uniquely pinned down as

$$F(v) = \Pr(V \leq v) = \Pr(\xi(B; G) \leq v) = G(\xi^{-1}(v; G)), \quad v \in [\underline{\xi}, \bar{\xi}].$$

**Sufficiency.** Conversely, if  $\xi(\cdot; G)$  is continuous and strictly increasing on  $(\underline{b}, \bar{b})$ , then  $G$  is rationalizable: define  $F$  on  $[\underline{\xi}, \bar{\xi}]$  as the pushforward of  $G$  through  $\xi(\cdot; G)$ ,

$$F(v) := \Pr(\xi(B; G) \leq v) = G(\xi^{-1}(v; G)), \quad v \in [\underline{\xi}, \bar{\xi}], \quad B \sim G, \quad (\text{F.5})$$

and define the candidate equilibrium strategy

$$s(v) := \xi^{-1}(v; G), \quad v \in [\underline{\xi}, \bar{\xi}]. \quad (\text{F.6})$$

Then  $s$  is an SMBNE for i.i.d. types  $V \sim F$ , and the induced equilibrium action distribution equals  $G$ .

**Proof for Sufficiency:** Assume now that  $\xi(\cdot; G)$  is continuous and strictly increasing on  $(\underline{b}, \bar{b})$ . Define  $F$  by (F.5) on  $[\underline{\xi}, \bar{\xi}]$  and define  $s(v) := \xi^{-1}(v; G)$  as in (F.6).

**Step 1 (The inverse of  $\xi$  reproduces exactly  $G$ ):** Let  $V \sim F$  and set  $B := s(V) = \xi^{-1}(V; G)$ . Then for any  $b \in [\underline{b}, \bar{b}]$ ,

$$\begin{aligned} \Pr(B \leq b) &= \Pr(\xi^{-1}(V; G) \leq b) = \Pr(V \leq \xi(b; G)) = F(\xi(b; G)) \\ &= G(\xi^{-1}(\xi(b; G); G)) = G(b), \end{aligned}$$

where the fourth equality uses (F.5) evaluated at  $v = \xi(b; G)$  and the fifth uses  $\xi^{-1}(\xi(b; G); G) = b$ . Since  $(V_1, \dots, V_N)$  are i.i.d.  $F$  and  $B_i = s(V_i)$  is applied componentwise,  $(B_1, \dots, B_N)$  are i.i.d. with common CDF  $G$ , so (C1) holds.

**Step 2 ( $s$  satisfies the interior first-order condition under beliefs  $G$ ):** Fix  $v \in (\underline{\xi}, \bar{\xi})$  and suppose the bidder believes  $B_{-i}$  are i.i.d. with CDF  $G$ . Then the bidder evaluates  $U(b, v; G) = vP(b; G) - T(b; G)$ , whose derivative in  $b$  is

$$\frac{\partial}{\partial b} U(b, v; G) = vP'(b; G) - T'(b; G).$$

At  $b = s(v) = \xi^{-1}(v; G)$  we have  $\xi(b; G) = v$ , so  $T'(b; G) = \xi(b; G)P'(b; G) = vP'(b; G)$  and therefore

$$\frac{\partial}{\partial b} U(s(v), v; G) = 0.$$

**Step 3 (Global optimality check):** Fix  $v \in (\underline{\xi}, \bar{\xi})$  and let  $b^* := s(v)$ . For any  $b \in [\underline{b}, \bar{b}]$ ,

$$U(b, v; G) - U(b^*, v; G) = \int_{b^*}^b (vP'(t; G) - T'(t; G)) dt = \int_{b^*}^b (v - \xi(t; G))P'(t; G) dt,$$

where the second line uses  $T'(t; G) = \xi(t; G)P'(t; G)$ . Since  $P'(t; G) > 0$  and  $\xi(\cdot; G)$  is strictly increasing, the integrand is strictly positive for  $t < b^*$  and strictly negative for  $t > b^*$ , implying  $U(b, v; G) \leq U(b^*, v; G)$  for all  $b$ , with strict inequality for  $b \neq b^*$  in the interior. Therefore  $b^* = s(v)$  is a global maximizer of  $U(\cdot, v; G)$  on  $[\underline{b}, \bar{b}]$ .

Thus, when types are i.i.d.  $F$  and all bidders play  $s$ , the induced action distribution is  $G$  and each type best-responds given beliefs  $G$ , so  $(F, s)$  rationalizes  $G$ .  $\square$

**Proof of Theorem 2.2:** Let

$$\sigma_{\nu, \epsilon}^2(b_1, b_2, q) = \max\{\sigma_{\nu}^2(b_1, b_2, q), \epsilon \cdot \sigma_{\nu}^2(\underline{b}, (\underline{b} + \bar{b})/2, 2)\}. \quad (\text{F.7})$$

Assumption 2.7 implies that  $\sup_{(b_1, b_2, q) \in \mathcal{L}} |\widehat{\sigma}_{\nu, \epsilon}^2(b_1, b_2, q) - \sigma_{\nu, \epsilon}^2(b_1, b_2, q)| \xrightarrow{p} 0$  and  $\sigma_{\nu, \epsilon}^2(b_1, b_2, q)$  is uniformly bounded away from zero for all  $(b_1, b_2, q) \in \mathcal{L}$ . Assumption 2.6 and Assumption

2.7 together imply that

$$\sqrt{S} \left( \frac{\widehat{\nu}(\cdot, \cdot, \cdot)}{\widehat{\sigma}_{\nu, \epsilon}(\cdot, \cdot, \cdot)} - \frac{\nu(\cdot, \cdot, \cdot)}{\sigma_{\nu, \epsilon}(\cdot, \cdot, \cdot)} \right) \Rightarrow \Phi_{\kappa_\epsilon}(\cdot, \cdot, \cdot)$$

where  $\kappa_\epsilon(\cdot, \cdot, \cdot) = \frac{\kappa((b'_1, b'_2, q'), (b''_1, b''_2, q''))}{\sigma_{\nu, \epsilon}(b'_1, b'_2, q') \sigma_{\nu, \epsilon}(b''_1, b''_2, q'')}$  for  $(b'_1, b'_2, q'), (b''_1, b''_2, q'') \in \mathcal{L}$ . Let  $\mathcal{L}^o \equiv \{(b_1, b_2, q) \in \mathcal{L} : \nu(b_1, b_2, q) = 0\}$  which is the collection of moment conditions hold with equality. Let  $\mathcal{L}^+ \equiv \{(b_1, b_2, q) \in \mathcal{L} : \nu(b_1, b_2, q) > 0\}$  which is the collection of strictly positive moment conditions. Under the null,  $\mathcal{L}^+$  is an empty set and under the alternative,  $\mathcal{L}^+$  contains at least one element. By a similar proof of Lemma 2.1 of Donald and Hsu (2016), under the null hypothesis, one can show that

$$\begin{aligned} \widehat{T}_S &= \sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \sqrt{S} \frac{\widehat{\nu}(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)}, 0 \right\}^2 Q(b_1, b_2, q) \\ &\xrightarrow{D} \sum_{(b_1, b_2, q) \in \mathcal{L}^o} \max \left\{ \Phi_{\kappa_\epsilon}^2(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q), \end{aligned} \quad (\text{F.8})$$

where  $\xrightarrow{D}$  denotes convergence in distribution. (F.8) shows that the limiting null distribution depends only on those moment conditions holding with equality. If under null hypothesis,  $\mathcal{L}^o$  is an empty set in that every moment condition holds with strictly inequality, then  $\widehat{T}_S$  converges in probability to zero at the rate  $S$ . Also, if under null hypothesis,  $\mathcal{L}^o$  is not an empty set, but  $\sigma_\nu(b_1, b_2, q) = 0$  for all  $(b_1, b_2, q) \in \mathcal{L}^o$ , then it is true that  $\widehat{T}_S$  converges in probability to zero at the rate  $S$  as well. These two cases are the degenerate in that the limiting null distribution converge in probability to a zero point. On the other hand, one can also show that  $\widehat{T}_S \rightarrow \infty$  at the rate of  $S$ . Next, similar to the proof of Lemma A.2 of Donald and Hsu (2016), one can show that the simulated test statistic defined as

$$\sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi^*(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} + \psi_S(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q)$$

converge in distribution to

$$\sum_{(b_1, b_2, q) \in \mathcal{L}} \max \left\{ \frac{\Phi^*(b_1, b_2, q)}{\widehat{\sigma}_{\nu, \epsilon}(b_1, b_2, q)} + \psi_S(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q)$$

$$\xrightarrow{D} \sum_{(b_1, b_2, q) \in \mathcal{L}^o \cup \mathcal{L}^+} \max \left\{ \Phi_{\kappa_\epsilon}^2(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q)$$

conditional on sample path in probability approaching one. Therefore, it follows that  $\hat{c}_\eta \xrightarrow{p} c(1 - \alpha + \eta) + \eta$ , where  $c(1 - \alpha + \eta)$  denotes the  $(1 - \alpha + \eta)$ -th quantile of

$$\sum_{(b_1, b_2, q) \in \mathcal{L}^o \cup \mathcal{L}^+} \max \left\{ \Phi_{\kappa_\epsilon}^2(b_1, b_2, q), 0 \right\}^2 Q(b_1, b_2, q)$$

and it is bounded in probability and strictly greater than  $\eta$ .

Then it is true that under the null, the asymptotic size of our test is  $\lim_{S \rightarrow \infty} P(\hat{T}_S \leq \hat{c}_\eta) \leq \alpha$ . Also, under the alternative, our test is consistent in that  $\lim_{S \rightarrow \infty} P(\hat{T}_S \leq \hat{c}_\eta) = 1$ .  $\square$

## G Proofs for Lemmas

**Proof of Lemma 2.1:** Fix any  $v \in (\underline{v}, \bar{v})$ . Under Assumption 2.2, the interim payoff,  $\Pi(b, v) = vP(b) - T(b)$ , is twice continuously differentiable in  $b$  on the relevant support. Assumption 2.4 states that

$$\Pi_{bb}(b, v) = \frac{\partial^2 \Pi(b, v)}{\partial b^2} = vP''(b) - T''(b) < 0,$$

so  $\Pi(\cdot, v)$  is strictly concave in  $b$  and hence has a unique maximizer. On any interval where the equilibrium best response is interior, the equilibrium strategy  $s(v)$  therefore satisfies the first-order condition

$$\Pi_b(s(v), v) = \left. \frac{\partial \Pi(b, v)}{\partial b} \right|_{b=s(v)} = vP'(s(v)) - T'(s(v)) = 0. \quad (\text{G.1})$$

Define  $\Pi_b(b, v) = vP'(b) - T'(b)$ . Then (G.1) can be written as  $\Pi_b(s(v), v) = 0$ . By Assumption 2.2,  $\Pi_b$  is continuously differentiable and  $\Pi_{bb}(b, v) < 0$  by Assumption 2.4.

and Assumption 2.3 states that  $\Pi_{bv}(b, v) = P'(b) > 0$ . Hence, by the implicit function

theorem,  $s(\cdot)$  is differentiable on any interior interval and

$$s'(v) = -\frac{P'(s(v))}{vP''(s(v)) - T''(s(v))} > 0. \quad (\text{G.2})$$

This completes the proof for Lemma 2.1.  $\square$

**Proof of Lemma 2.2:** We apply Lemma 3.1 of Hsu and Shen (2021) to show this lemma. We set the  $\lambda(b)$  and  $h(b)$  functions of Lemma 3.1 of Hsu and Shen (2021) as  $T'(b)/P'(b)$  and  $h(b)P'(b)$  in our example. Then it follows that

$$\begin{aligned} M(b, q) &\equiv \int_b^{b+\frac{a}{q}} \frac{T'(b)}{P'(b)} \cdot h(b)P'(b)db = \int_b^{b+\frac{a}{q}} T'(b)h(b)db, \text{ and} \\ W(b, q) &\equiv \int_b^{b+\frac{a}{q}} h(b)P'(b)db. \end{aligned}$$

This shows Lemma 2.2.  $\square$

**Proof of Lemma 3.1:** It is sufficient to show that  $M(b, q)$  and  $W(b, q)$  are identified as (3.6) and (3.5). Following the discussion after (3.3), we have

$$\begin{aligned} M(b, q) &= \int_b^{b+\frac{a}{q}} (\tilde{b}g(\tilde{b}) + (N-1)^{-1}G(\tilde{b}))d\tilde{b} \\ &= E\left[B_{i,\ell}1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right] \\ &\quad + \frac{1}{N-1}E\left[1\left(B_{i,\ell} \leq b + \frac{a}{q}\right)\left(b + \frac{a}{q} - B_{i,\ell}\right) - 1(B_{i,\ell} \leq b)(b - B_{i,\ell})\right], \\ W(b, q) &= \int_b^{b+\frac{a}{q}} g(\tilde{b})d\tilde{b} = E\left[1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right]. \end{aligned}$$

Note that the second equality holds because  $\int_b^b G(\tilde{b})d\tilde{b} = E[1(B_{i,\ell} \leq \tilde{b})(\tilde{b} - B_{i,\ell})]$ .  $\square$

**Proof of Lemma 3.2** From the discussion after (3.10), it is sufficient to show that

$$\begin{aligned} M(b, q) &= \int_b^{b+\frac{a}{q}} \tilde{b}g(\tilde{b})d\tilde{b} = E\left[B_{i,\ell} \cdot 1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\right], \\ W(b, q) &= E_{B_{-i}}[D(\tilde{b}, B_{-i})(1 - D(\tilde{b}, B_{-i}))g(\tilde{b})d\tilde{b}] \\ &= E\left[1\left(b \leq B_{i,\ell} \leq b + \frac{a}{q}\right)\frac{B_{i,\ell}}{\sum_{j=1}^N B_{j,\ell}}\left(1 - \frac{B_{i,\ell}}{\sum_{j=1}^N B_{j,\ell}}\right)\right], \end{aligned}$$

where the last equality holds by law of iterated expectations.  $\square$

The proofs for Lemmas 3.3 and 3.4 are similar to that for Lemma 3.2, so we omit the details.

**Proof of Lemma 4.1:** We apply Lemma A.1 of Hsu and Shen (2021) to show Lemma 4.1. Similar to proof for Lemma 2.2, we set  $\lambda(b, x)$  and  $h(b, x)$  functions in Lemma A.1 of Hsu and Shen (2021) as  $T'(b, x)/P'(b, x)$  and  $h(b, x)P'(b, x)$  in our example. Then it follows that

$$M(b, x, q) \equiv \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{a}{q}} \frac{T'(\tilde{b}, \tilde{x})}{P'(\tilde{b}, \tilde{x})} \cdot h(\tilde{b}, \tilde{x})P'(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x} = \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{a}{q}} T'(\tilde{b}, \tilde{x})h(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x}, \text{ and}$$

$$W(b, x, q) \equiv \int_x^{x+\frac{1}{q}} \int_b^{b+\frac{a}{q}} h(\tilde{b}, \tilde{x})P'(\tilde{b}, \tilde{x})d\tilde{b}d\tilde{x}.$$

This shows Lemma 4.1.  $\square$

## H Additional Simulation Results

We consider the case with heterogeneous number of bidders and for implementation of such test; see Appendix B for details. To study the size and power properties, we use the same quantile function of the true bid distribution defined in (5.2). We set  $k = 0.5$  for the size analysis and set 10, 20, and 40 for the power analysis. For all values of  $k$ , we set  $N_t = [2, 3, 4]$  and  $L_t = a[60, 40, 20]$  with  $a = 1, 2, 3$ . Figure 2 presents the quasi-inverse equilibrium strategy  $\xi(b)$  for different values of  $k$  and  $N$ .

Table 6 shows the rejection probabilities of our tests for the case with heterogeneous number of bidders. Like the results in Table 1, the proposed test controls size well for  $k = 0.5$ , and the power increases with the sample size and with  $k$  for  $k \geq 10$ . Table 6 also shows that the choices of  $q_1$  do not affect the test performance much.

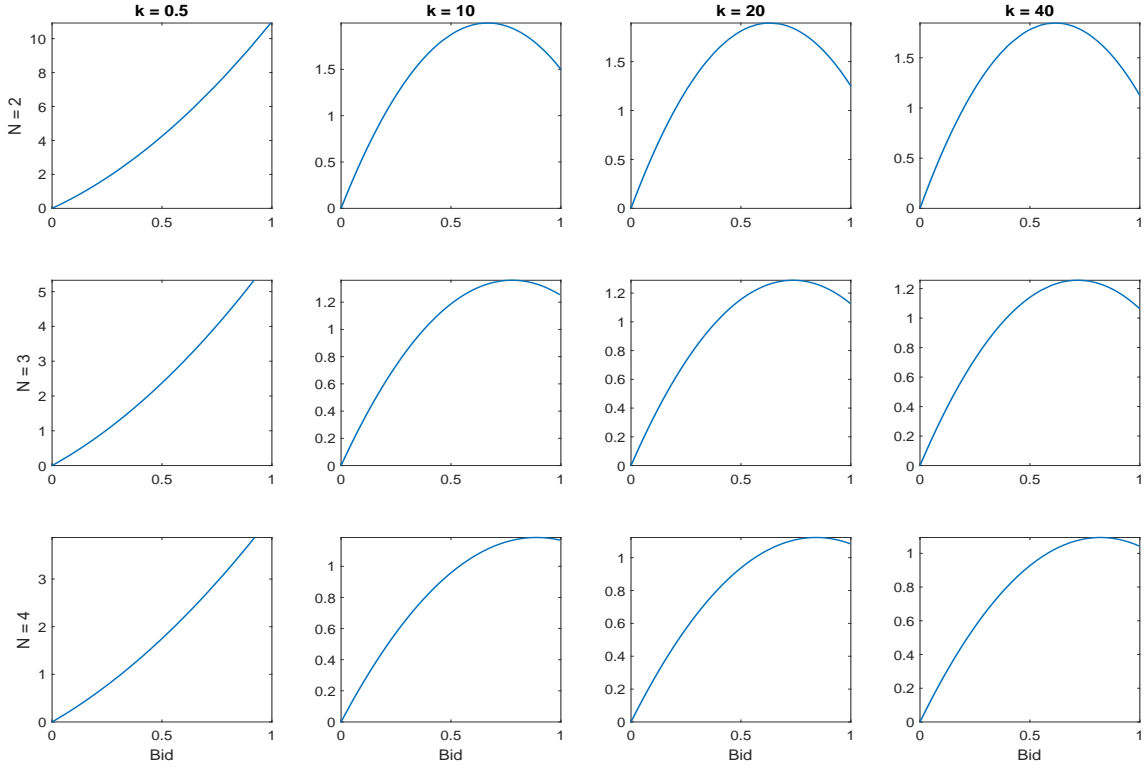


Figure 2: Quasi-inverse equilibrium strategy for  $N = 2, 3,$  or  $4$

Table 6: Rejection probabilities for the case without covaraites and with heterogeneous number of bidders

k	a	$n_c=15$	$n_c=20$	$n_c=25$	$n_c=30$
0.5	1	0.000	0.000	0.000	0.000
0.5	2	0.000	0.000	0.000	0.000
0.5	3	0.000	0.000	0.000	0.000
10	1	0.195	0.204	0.175	0.166
10	2	0.209	0.227	0.225	0.229
10	3	0.298	0.296	0.275	0.298
20	1	0.449	0.413	0.442	0.369
20	2	0.669	0.640	0.629	0.618
20	3	0.800	0.800	0.796	0.805
40	1	0.723	0.674	0.659	0.616
40	2	0.922	0.924	0.920	0.893
40	3	0.989	0.988	0.987	0.982