Monotonicity Test for Local Average Treatment Effects Under Regression Discontinuity

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Abstract

This paper proposes nonparametric monotonicity tests for the (local) average treatment effects under the sharp and fuzzy regression discontinuity designs. The tests allow researchers to examine whether the policy effect of interest has monotonic relationships with conditioning covariates. We show the consistency and asymptotic uniform size control of the proposed tests. The proposed tests are applied to re-investigate the impact of attending a better high school using the Romanian data set studied in Pop-Eleches and Urquiola (2013). We find that the effect of going to a better school on a student's probability of taking the Baccalaureate exam increases monotonically with peer quality of the school.

Keywords: average treatment effect, local average treatment effect, regression discontinuity, regression monotonicity, nonparametric

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1 Introduction

Regression monotonicity is an important and well-studied topic in econometrics. Economic theory in fields of microeconomics such as industrial organization, labor, trade often give predictions of monotonicity and applied researchers are interested in testing such predictions with data. Previous works in testing regression monotonicity include Ghosal et al. (2000), Hall and Heckman (2000), Chetverikov (2013), and Hsu et al. (2017), among many others, and please see Chetverikov (2013) for a more complete literature review. Unfortunately, none of the aforementioned tests could be directly applied to the regression discontinuity (RD) framework, which has become very popular in applied microeconomics in the recent years. Therefore, in this paper, we propose monotonicity tests under the RD set-up. Specifically, we propose tests that allow practitioners to investigate whether the average treatment effect (ATE) or local average treatment effect (LATE) of a policy or treatment intervention has certain monotonic relationship with pre-determined covariates.

The RD framework refers to econometric models where the probability of an individual receiving a policy or treatment intervention changes discontinuously at a cut-off point of an underlying variable. Researchers compare the response outcomes above and below the cut-off point so as to identify the treatment effect of individuals at the margin of treatment intervention. Pioneering works and early applications in RD include van der Klaauw (2002), Angrist and Lavy (1999), and Black (1999), among others. Imbens and Lemieux (2008) and Lee and Lemieux (2010) provide excellent reviews.

The tests we propose are important to applied researchers who investigate treatment effects in RD analysis. Hsu and Shen (2016) conduct a brief survey of recent publications in top journals in economics and find that over 85% of the empirical RD papers analyze treatment effect heterogeneity. For example, Ito (2015) uses a sharp RD design to study the impact of an electricity rebate program in California. Using a parametric model that interacts the treatment dummy with climate conditions and income levels, Ito (2015) finds that the treatment effect of the rebate program monotonically increases with the average temperature and decreases with income levels. However, the interaction term method used in Ito (2015) is parametric and may generate empirical findings subject to model misspecification. Instead, the monotonicity tests we propose are fully nonparametric and this is in line with the nonparametric identification and estimation strategy in classic RD models.

The proposed tests contribute to the regression monotonicity literature as well. Since the identification of treatment effects in the RD framework relies on discontinuity points in the treatment take-up decision, boundary local linear estimators are often used in estimation. However, all existing monotonicity tests in the regression monotonicity literature, to the best of our knowledge, only consider parametric estimators or standard interior estimators. Therefore, it is of theoretical interest to propose monotonicity tests that are suitable for the RD framework.

The instrumental function approach we adopt is related to the Andrews and Shi (2013, 2015) and Hsu et al. (2017). Andrews and Shi (2013, 2015) propose to use it to transform a finite number of conditional moment inequalities to an infinite number of unconditional moment inequalities without loss of information and build test statistics upon the latter. Hsu et al. (2017) generalize the method to test for generalized regression monotonicity. In this paper, we adopt the generalized instrumental function approach of Hsu et al. (2017) to develop monotonicity tests for the average policy effects under both sharp and fuzzy RD designs.

Our tests have several nice features. First, the proposed tests do not require nonparametric derivative estimation. Second, the proposed test statistics are of order $(nh)^{-1/2}$, meaning that although we are looking at policy effects conditioning on additional control variables, the test statistics have the same rate of convergence as the classic mean RD estimators that do not control covariates other than the running variable. Third, our test is robust to weak identification under the fuzzy RD set-up. This is particularly important in heterogeneity analysis of the LATE because first stage heterogeneity can easily result in low first stage complying rate for some subpopulation of interest even though the overall identification for the whole population is strong. Moreover, to investiage whether the LATE is monotonic with some other covariate, for Last, the proposed tests have uniform size control over a broad set of data generating processes.

Monte Carlo experiments show that the proposed tests have very good size and power properties in small sample. We apply the tests to study the impact of attending a better high school in Romania following Pop-Eleches and Urquiola (2013). Mean RD analysis in Pop-Eleches and Urquiola (2013) finds that going to a more selective high school significantly improves the average grade among marginal students in the Baccalaureate exam but does not seem to affect the probability of a student taking the Baccalaureate exam. In contrast, our monotonicity test reveals that the effect on exam taking rate increases monotonically with peer quality of the more selective high school, indicating that the insignificant mean effect found in Pop-Eleches and Urquiola (2013) is due to cancelation of positive and negative treatment effects among schools with different peer qualities.

The paper is organized as follows. Section 2 sets up the RD model and proposes the benchmark monotonicity test for the general fuzzy RD case. Section 3 discusses the asymptotic property of the proposed test. Section 4 extends the benchmark test to the special case of sharp RD. Section 5 examines the small sample performance of the proposed tests. Section 6 applies the proposed tests to re-examine the effect of going to a better school using the Romanian dataset published by Pop-Eleches and Urquiola (2013). Proofs of all lemmas and theorems are provided in the Appendix.

2 Testing Treatment Effect Monotonicity Under the Fuzzy RD Design

2.1 Model Set-up and Null Hypothesis

Let Y denote the outcome of interest, and T denote the treatment status of an individual. Use $Y(0)$ and $Y(1)$ to denote potential outcomes when $T = 0$ and $T = 1$, respectively. The observed outcome $Y = T \cdot Y(1) + (1-T) \cdot Y(0)$. Whether an individual receives treatment or not depends at least partially on the running variable Z. A policy intervention encourages an individual to receive treatment if the running variable Z is larger than or equal to some cut-off value c. Let $T(1)$ and $T(0)$ be the potential treatment decisions of an individual depending on whether he/she is encouraged or not. T, $T(0)$, and $T(1)$ are all binary indicators. The observed treatment status $T = T(1)1(Z \geq c) + T(0)1(Z < c)$. Let X be a set of covariates with compact support $\mathcal{X} \subset R^{d_x}$. Without loss of generality, we assume that $\mathcal{X} = \times_{j=1}^{d_x}[0,1]$. For notational simplicity, we assume that X includes only continuous variables. Our results could be easily extended to cases where X includes discrete variables.

The RD model follows a sharp design if the treatment decision T is a deterministic function of the running variable Z and a fuzzy design if T is a probabilistic function of Z. We focus on the more general case of fuzzy RD design in this section and leave the sharp RD design to Section 4.

Let (Z, T, X, Y) be a vector of random variables with underlying distribution P. E_P denotes the expectation under distribution P. Use $\mathcal{X}_z \subseteq \mathcal{X}$ to denote the support of X conditional on $Z = z$. For any $\delta > 0$, let $\mathcal{N}_{\delta,z}(c) = \{z : |z - c| \leq \delta\}$ denote a neighborhood of Z around $Z = c$. The following assumption collects the identifying conditions for the LATE and the conditional local average treatment effect (CLATE).

Assumption 2.1 *For a running variable Z continuously distributed in $\mathcal{N}_{\delta,z}(c)$ for some $\delta > 0$, assume that*

(i) $E_P[Y(t)|T(1)-T(0) = 1, X = x, Z = z]$ *and* $E_P[Y(t)|T(1) = T(0) = t', X = x, Z = z]$ *are continuous in x and z on $\mathcal{X}_c \times \mathcal{N}_{\delta,z}(c)$ for $t, t' \in \{0, 1\}$;*

(ii) $E_P[T(1) - T(0) = 1|X = x, Z = z]$ *and* $E_P[T(1) = T(0) = t|X = x, Z = z]$ *are continuous in x and z on $\mathcal{X}_c \times \mathcal{N}_{\delta,z}(c)$ for $t \in \{0, 1\}$;*

(iii) $T(1) \geq T(0)$;

(iv) $E_P[T(1) - T(0)|X = x, Z = c] > 0$ *for all $x \in \mathcal{X}_c$;*

(v) The distribution function of $X|Z = z$ is continuous in z on $\mathcal{N}_{\delta,z}(c)$.

Assumption 2.1(i) requires the continuity of average potential outcomes for always-taker (individuals with $T(0) = T(1) = 1$), compliers (individuals with $T(1) - T(0) = 1$), and never-takes (individuals with $T(0) = T(1) = 0$) along both the running variable Z and the additional control X. Assumption 2.1(ii) requires that the proportion of each group is continuous along both Z and X. Assumption 2.1(iii) and (iv) require no defiers and non-trivial presence of compliers, respectively. Assumption 2.1(i), (ii) and (iv) are stronger than their counterparts that are unconditional on X (c.f. Dong and Lewbel, 2015) as we are interested in identifying of the CLATE conditional on the additional covariate X. Assumption 2.1(v) requires that the conditional distribution of the additional controls conditional on the running variable is continuous. It is a direct implication of the "no precise control over the running variable" rule introduced in Lee and Lemieux (2010) and well-accepted in the applied RD literature. This condition is also required to identify the CLATE.

Under Assumption 2.1, LATE and CLATE are identified by the following expressions.

$$\begin{aligned}
LATE &= E_P[Y(1) - Y(0)|Z = c, T(1) - T(0) = 1] \\
&= \frac{E_P[(Y(1) - Y(0))\,(T(1) - T(0))\,|Z = c]}{E_P[T(1) - T(0)|Z = c]} \\
&= \frac{\lim_{z\searrow c} E_P[Y|Z = z] - \lim_{z\nearrow c} E_P[Y|Z = z]}{\lim_{z\searrow c} E_P[T|Z = z] - \lim_{z\nearrow c} E_P[T|Z = z]}, \\
CLATE(x) &= E_P[Y(1) - Y(0)|X = x, Z = c, T(1) - T(0) = 1], \\
&= \frac{E_P[(Y(1) - Y(0))\,(T(1) - T(0))\,|X = x, Z = c]}{E_P[T(1) - T(0)|X = x, Z = c]} \\
&= \frac{\lim_{z\searrow c} E_P[Y|X = x, Z = z] - \lim_{z\nearrow c} E_P[Y|X = x, Z = z]}{\lim_{z\searrow c} E_P[T|X = x, Z = z] - \lim_{z\nearrow c} E_P[T|X = x, Z = z]}.
\end{aligned}$$

The dependence of LATE and CLATE(x) on P is suppressed for notational simplicity. Proofs of the identification results are given in Hsu and Shen (2016).

The null hypothesis of interest is whether CLATE monotonically increases with some elements in X. Partition X such that $X = (W, S)$ with $W \in \mathcal{W} = [0, 1]^{d_w}$ and $S \in \mathcal{S} = [0, 1]^{d_s}$; $d_w \geq 1$, $d_s \geq 0$ and $d_s + d_w = d$. Specifically, we are interested in the following null and alternative hypotheses.

$$\begin{aligned}
&H_{0,FRD} : CLATE(x) \text{ is non-decreasing in } w \text{ on } \mathcal{W} \text{ for all } s \in \mathcal{S}; \qquad (2.1)\\
&H_{1,FRD} :\ H_{0,FRD} \text{ does not hold.}
\end{aligned}$$

To test the null hypothesis $H_{0,FRD}$, a direct approach is to take partial derivative of $CLATE(x)$ with respect to w and examine the sign of the derivative function for all values of w and s. This direct approach requires both the numerator and the denominator of $CLATE(x)$ to be differentiable with respect to w. It also requires the use of nonparametric derivative estimation, which has a slow convergence rate. In this paper, we take an alternative route. We follow the idea of instrumental function approach in Hsu et al.

(2017) and transform the null hypothesis $H_{0,FRD}$ to a series of countably many moment inequalities that do not involve derivatives. The proposed test based on the transformed null hypothesis does not require the use of nonparametric derivative estimators and has test statistics with convergence rate $(nh)^{-1/2}$, regardless of the dimension of the conditioning covariates. Further, the null hypothesis transformation avoids the use of fractions in the testable implications. Therefore, the test statistic we use does not involve random denominators and our test is robust to weak identification.

2.2 Transformation of the Null Hypothesis

This section discusses the transformation of the null hypothesis $H_{0,FRD}$. For any $w_1, w_2 \in \mathcal{W}$, $s \in \mathcal{S}$, and $q \in \mathcal{Z}_+$, where $\mathcal{Z}_+$ denotes the set of positive intergers, define

$$C_{w_1,q} \equiv \left(\prod_{j=1}^{d_w}\left[w_{1j}, w_{1j} + \frac{1}{q+1}\right]\right) \cap \mathcal{W}, \quad C_{w_2,q} \equiv \left(\prod_{j=1}^{d_w}\left[w_{2j}, w_{2j} + \frac{1}{q+1}\right]\right) \cap \mathcal{W},$$

$$C_{s,q} \equiv \left(\prod_{j=1}^{d_s}\left[s_j, s_j + \frac{1}{q}\right]\right) \cap \mathcal{S},$$

where w_{1j}, w_{2j}, and s_j are the j-th dimension of w_1, w_2 and s. Also, when $d_w \geq 2$, denote $w_1 \geq w_2$ iff $w_{1j} \geq w_{2j}$ for all $j = 1, \ldots, d_w$; denote $w_1 > w_2$ iff $w_{1j} \geq w_{2j}$ for all $j = 1, \ldots, d_w$, and $w_{1k} > w_{2k}$ for some $k \in \{1, \ldots, d_w\}$; denote $w_1 \gg w_2$ iff $w_{1j} > w_{2j}$ for all $j = 1, \ldots, d_w$.

Lemma 2.1 *Let $\lambda(w, s)$ be a continuous function in (w, s) on $\mathcal{W} \times \mathcal{S} = [0,1]^{d_w+d_s}$, and $0 < h(w, s) < M < \infty$ be a weight function. The following two statements are equivalent:*

(i) $\lambda(w_1, s) \geq \lambda(w_2, s)$ whenever $w_1 \geq w_2$ for any $w_1, w_2 \in \mathcal{W}$ and for any $s \in \mathcal{S}$;

(ii) for any $q \in \mathcal{Z}_+$, and $w_1 \geq w_2$ such that $(q+1) \cdot w_1, (q+1) \cdot w_2 \in \{0, 1, 2, \cdots, q\}^{d_w}$ and $q \cdot s \in \{0, 1, 2, \ldots, q-1\}^{d_s}$,

$$\frac{\int_{C_{w_1,q} \times C_{s,q}} \lambda(\tilde{w}, \tilde{s}) \cdot h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}}{\int_{C_{w_1,q} \times C_{s,q}} h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}} \geq \frac{\int_{C_{w_2,q} \times C_{s,q}} \lambda(\tilde{w}, \tilde{s}) \cdot h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}}{\int_{C_{w_2,q} \times C_{s,q}} h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}}.$$

Lemma 2.1 states that the monotonicity condition of any continuous function could be re-formulated as a countable number of moment inequalities. When $\lambda(w, s) = CLATE(w, s)$, statement (i) in Lemma 2.1 reduces to the null hypothesis $H_{0,FRD}$. Let the weight function in (ii) be $h(w,s) = E_P[T(1)-T(0)|W=w, S=s, Z=c] \cdot f_{W,S|Z=c}(w,s)$, then inequality in Lemma 2.1.(ii) reduces to

$$
\begin{aligned}
&\frac{E_P[g_{w_1,\ell}(W)g_{s,\ell}(S)(Y(1)-Y(0))(T(1)-T(0))|Z=c]}{E_P[g_{w_1,\ell}(W)g_{s,\ell}(S)(T(1)-T(0))|Z=c]}\\
\geq&\frac{E_P[g_{w_2,\ell}(W)g_{s,\ell}(S)(Y(1)-Y(0))(T(1)-T(0))|Z=c]}{E_P[g_{w_2,\ell}(W)g_{s,\ell}(S)(T(1)-T(0))|Z=c]},
\end{aligned}
$$

where $g_{w_\kappa,\ell}(w) = 1(w \in C_{w_\kappa,q})$ for $\kappa = 1,2$, $g_{s,\ell}(s) = 1(s \in C_{s,q})$, and $\ell = (w_1, w_2, s, q) \in \mathcal{W}^2 \times \mathcal{S} \times \mathcal{Z}_+$. By Assumption 2.1 and standard derivations, we can further replace the expectations in the above inequality with expressions that only involve observed variables, since for all $\ell \in \mathcal{W}^2 \times \mathcal{S} \times \mathcal{Z}_+$, and $\kappa = 1,2$,

$$
\begin{aligned}
&E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)(Y(1)-Y(0))(T(1)-T(0))|Z=c]\\
=&\lim_{z\searrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)Y|Z=z] - \lim_{z\nearrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)Y|Z=z], \text{ and}\\
&E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)(T(1)-T(0))|Z=c]\\
=&\lim_{z\searrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)T|Z=z] - \lim_{z\nearrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)T|Z=z].
\end{aligned}
$$

Define $m_{P,+}^{(\kappa)}(\ell) = \lim_{z\searrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)Y|Z=z]$, $m_{P,-}^{(\kappa)}(\ell) = \lim_{z\nearrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)Y|Z=z]$, $q_{P,+}^{(\kappa)}(\ell) = \lim_{z\searrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)T|Z=z]$, $q_{P,-}^{(\kappa)}(\ell) = \lim_{z\nearrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)T|Z=z]$, $\rho_P^{(\kappa)}(\ell) = m_{P,+}^{(\kappa)}(\ell) - m_{P,-}^{(\kappa)}(\ell)$, and $\varrho_P^{(\kappa)}(\ell) = q_{P,+}^{(\kappa)}(\ell) - q_{P,-}^{(\kappa)}(\ell)$ for all $\ell \in \mathcal{W}^2 \times \mathcal{S} \times \mathcal{Z}_+$ and $\kappa = 1,2$. The following lemma summarizes the hypothesis transformation result discussed above.

Lemma 2.2 *Under Assumption 2.1, the null hypothesis $H_{0,FRD}$ is equivalent to*

$$
H'_{0,FRD} : \mu_P(\ell) \equiv \rho_P^{(2)}(\ell)\varrho_P^{(1)}(\ell) - \rho_P^{(1)}(\ell)\varrho_P^{(2)}(\ell) \leq 0, \quad \textit{for all } g_\ell \in \mathcal{G}, \tag{2.2}
$$

where $\mathcal{G} = \{g_\ell = (g_{w_1,\ell}, g_{w_2,\ell}, g_{s,\ell}) : \ell \in \mathcal{L}\}$ is a set of the indicator functions of countable hypercubes with

$$
\begin{aligned}
\mathcal{L} = \Big\{(w_1, w_2, s, q) :\ & (q+1)\cdot(w_1,w_2) \in \{0,1,2,\cdots,q\}^{2d_w}, w_1 \geq w_2,\\
& q \cdot s \in \{0,1,2,\cdots,q-1\}^{d_s},\ \textit{for } q = 1,2,3\cdots\Big\}.
\end{aligned} \tag{2.3}
$$

Note that the inequality in (2.2) also avoids the use of fraction terms that appeared in the inequality in Lemma 2.1. Monotonicity test built upon this null transformation is therefore robust to weak identification under the fuzzy RD set-up. As is discussed in the introduction, it is important to have the proposed test robust to weak identification because first stage heterogeneity can easily result in weak identification of $CLATE(x)$ for some values of x even if the identification of $LATE$ for the whole population is strong.

2.3 Test Statistic

Let $\{Z_i, T_i, X_i, Y_i\}_{i=1}^n$ be a sample of size n randomly drawn from the underlying distribution of (Z, T, X, Y). In the following, we introduce the nonparametric local linear estimator of the moment function $\mu_P(\ell)$ defined in equation (2.2), and then the test statistic for testing $H'_{0,FRD}$, or equivalently $H_{0,FRD}$.

For all $\ell \in \mathcal{W}^2 \times \mathcal{S} \times \mathcal{Z}_+$ and $\kappa = 1, 2$, let $\hat{m}^{(\kappa)}_{n,+}(\ell)$, $\hat{m}^{(\kappa)}_{n,-}(\ell)$, $\hat{p}^{(\kappa)}_{n,+}(\ell)$ and $\hat{p}^{(\kappa)}_{n,-}(\ell)$ be the local linear estimators of $m^{(\kappa)}_{P,+}(\ell)$, $m^{(\kappa)}_{P,-}(\ell)$, $p^{(\kappa)}_{P,+}(\ell)$ and $p^{(\kappa)}_{P,-}(\ell)$. Let $K(.)$ be the symmetric kernel function and h the bandwidth; estimators $\hat{m}^{(\kappa)}_{n,+}(\ell)$, $\hat{m}^{(\kappa)}_{n,-}(\ell)$, $\hat{p}^{(\kappa)}_{n,+}(\ell)$ and $\hat{p}^{(\kappa)}_{n,-}(\ell)$ are the constant terms of the following minimization problems, respectively.

$$(\hat{m}^{(\kappa)}_{n,+}(\ell), \hat{a}^{(\kappa)}_{n,+}(\ell)) = \arg\min_{a,b} \sum_{Z_i \geq c}^{n} K\left(\frac{Z_i - c}{h}\right) \Big[g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) Y_i - a - b(Z_i - c)\Big]^2,$$

$$(\hat{m}^{(\kappa)}_{n,-}(\ell), \hat{a}^{(\kappa)}_{n,-}(\ell)) = \arg\min_{a,b} \sum_{Z_i < c}^{n} K\left(\frac{Z_i - c}{h}\right) \Big[g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) Y_i - a - b(Z_i - c)\Big]^2,$$

$$(\hat{q}^{(\kappa)}_{n,+}(\ell), \hat{b}^{(\kappa)}_{n,+}(\ell)) = \arg\min_{a,b} \sum_{Z_i \geq c}^{n} K\left(\frac{Z_i - c}{h}\right) \Big[g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) T_i - a - b(Z_i - c)\Big]^2,$$

$$(\hat{q}^{(\kappa)}_{n,-}(\ell), \hat{b}^{(\kappa)}_{n,-}(\ell)) = \arg\min_{a,b} \sum_{Z_i < c}^{n} K\left(\frac{Z_i - c}{h}\right) \Big[g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) T_i - a - b(Z_i - c)\Big]^2.$$

Following Fan and Gijbels (1992), for $l = 0, 1, 2, \ldots$, define

$$S^+_{n,l} = \sum_{i=1}^{n} 1(Z_i \geq c) K\left(\frac{Z_i - c}{h}\right)(Z_i - c)^l, \;\; S^-_{n,l} = \sum_{i=1}^{n} 1(Z_i < c) K\left(\frac{Z_i - c}{h}\right)(Z_i - c)^l,$$

$$\mathsf{w}^+_{ni} = \frac{1(Z_i \geq c) K(\frac{Z_i - c}{h})[S^+_{n,2} - S^+_{n,1}(Z_i - c)]}{S^+_{n,0} S^+_{n,2} - S^+_{n,1} S^+_{n,1}}, \;\; \mathsf{w}^-_{ni} = \frac{1(Z_i < c) K(\frac{Z_i - c}{h})[S^-_{n,2} - S^-_{n,1}(Z_i - c)]}{S^-_{n,0} S^-_{n,2} - S^-_{n,1} S^-_{n,1}}.$$

Then estimators $\hat{m}^{(\kappa)}_{n,+}(\ell)$, $\hat{m}^{(\kappa)}_{n,-}(\ell)$, $\hat{q}^{(\kappa)}_{n,+}(\ell)$ and $\hat{q}^{(\kappa)}_{n,-}(\ell)$ could be re-written as

$$\hat{m}^{(\kappa)}_{n,+}(\ell) = \sum_{i=1}^{n} \mathsf{w}^{+}_{ni} \cdot m^{(\kappa)}(Y_i, W_i, S_i, \ell), \quad \hat{m}^{(\kappa)}_{n,-}(\ell) = \sum_{i=1}^{n} \mathsf{w}^{-}_{ni} \cdot m^{(\kappa)}(Y_i, W_i, S_i, \ell),$$
$$\hat{q}^{(\kappa)}_{n,+}(\ell) = \sum_{i=1}^{n} \mathsf{w}^{+}_{ni} \cdot q^{(\kappa)}(T_i, W_i, S_i, \ell), \quad \hat{q}^{(\kappa)}_{n,-}(\ell) = \sum_{i=1}^{n} \mathsf{w}^{-}_{ni} \cdot q^{(\kappa)}(T_i, W_i, S_i, \ell).$$

where $m^{(\kappa)}(Y_i, W_i, S_i, \ell) = g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) Y_i$, and $q^{(\kappa)}(T_i, W_i, S_i, \ell) = g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) T_i$.

For all $\ell \in \mathcal{W}^2 \times \mathcal{S} \times \mathcal{Z}_+$ and $\kappa = 1, 2$, define $\hat{\rho}^{(\kappa)}_n(\ell) = \hat{m}^{(\kappa)}_{n,+}(\ell) - \hat{m}^{(\kappa)}_{n,-}(\ell)$ and $\hat{\varrho}^{(\kappa)}_n(\ell) = \hat{q}^{(\kappa)}_{n,+}(\ell) - \hat{q}^{(\kappa)}_{n,-}(\ell)$ as the estimators of $\rho^{(\kappa)}_P(\ell)$ and $\varrho^{(\kappa)}_P(\ell)$, respectively. We will show in Section 3 that under proper regularity conditions on the underlying data distribution and mild kernel and bandwidth conditions, $\hat{\rho}^{(\kappa)}_n$ and $\hat{\varrho}^{(\kappa)}_n$ are uniformly consistent over $\ell \in \mathcal{L}$. Further, define $\hat{\mu}_n(\ell) = \hat{\rho}^{(2)}_n(\ell)\hat{\varrho}^{(1)}_n(\ell) - \hat{\rho}^{(1)}_n(\ell)\hat{\varrho}^{(2)}_n(\ell)$ and

$$\phi^{(\kappa)}_{\rho,ni}(\ell) = \sqrt{nh}\left(\mathsf{w}^{+}_{ni}\left(m^{(\kappa)}(Y_i, W_i, S_i, \ell) - m^{(\kappa)}_{P,+}(\ell)\right) - \mathsf{w}^{-}_{ni}\left(m^{(\kappa)}(Y_i, W_i, S_i, \ell) - m^{(\kappa)}_{P,-}(\ell)\right)\right),$$
$$\phi^{(\kappa)}_{\varrho,ni}(\ell) = \sqrt{nh}\left(\mathsf{w}^{+}_{ni}\left(q^{(\kappa)}(T_i, W_i, S_i, \ell) - q^{(\kappa)}_{P,+}(\ell)\right) - \mathsf{w}^{-}_{ni}\left(q^{(\kappa)}(T_i, W_i, S_i, \ell) - q^{(\kappa)}_{P,-}(\ell)\right)\right).$$

Uniform consistency of $\hat{\rho}^{(\kappa)}_n$ and $\hat{\varrho}^{(\kappa)}_n$ also implies the following influence function representation of $\sqrt{nh}(\hat{\mu}_n(\ell) - \mu_P(\ell))$:

$$\begin{aligned}
&\sqrt{nh}(\hat{\mu}_n(\ell) - \mu_P(\ell)) \\
=& \sum_{i=1}^{n} \varrho^{(1)}_P(\ell) \cdot \phi^{(2)}_{\rho,ni}(\ell) + \rho^{(2)}_P(\ell) \cdot \phi^{(1)}_{\varrho,ni}(\ell) - \varrho^{(2)}_P(\ell) \cdot \phi^{(1)}_{\rho,ni}(\ell) - \rho^{(1)}_P(\ell) \cdot \phi^{(2)}_{\varrho,ni}(\ell) + o_p(1) \\
\equiv& \sum_{i=1}^{n} \phi_{\mu,ni}(\ell) + o_p(1),
\end{aligned}$$

with the $o_p(1)$ result holding uniformly over $\ell \in \mathcal{L}$. In Section 3, we will use this representation to obtain the asymptotic property of the local linear estimator $\hat{\mu}_n$ and the test statistic $\widehat{T}_{n,FRD}$ defined below.

For any $\ell \in \mathcal{L}$, let $\hat{\sigma}^2_{\mu,n}(\ell) = \sum_{i=1}^{n} \hat{\phi}_{\mu,ni}(\ell)^2$, where $\hat{\phi}_{\mu,ni}(\ell)$ is the estimated influence function with $\rho^{(\kappa)}_P(\ell)$ and $\varrho^{(\kappa)}_P(\ell)$ in $\phi_{\mu,ni}(\ell)$ replaced by $\hat{\rho}^{(\kappa)}_n(\ell)$ and $\hat{\varrho}^{(\kappa)}_n(\ell)$, respectively. In Lemma B.4 in the appendix, we show that under proper regularity conditions $\hat{\sigma}^2_{\mu,n}(\ell)$ is a consistent estimator for $\sigma^2_{\mu,P}(\ell)$, or the asymptotic variance of $\sqrt{nh}(\hat{\mu}_n(\ell) - \mu_P(\ell))$. Let ϵ be some small positive number and $\mathbf{a}$ be a vector of scalar a's. Define $\hat{\sigma}^2_{\mu,\epsilon}(\ell) = \max\left\{\hat{\sigma}^2_{\mu,n}(\ell), \epsilon \cdot \hat{\sigma}^2_{\mu,n}\left((\mathbf{1/2}, \mathbf{0}, \mathbf{0}, 1)\right)\right\}$ which manually bounds the variance estimator away from zero. To test the null hypothesis $H'_{0,FRD}$ described in the previous section, we use the Kolmogorov-Smirnov type statistic

$$\widehat{T}_{n,FRD} = \sup_{\ell \in \mathcal{L}} \sqrt{nh} \frac{\hat{\mu}_n(\ell)}{\hat{\sigma}_{\mu,\epsilon}(\ell)},$$

which converges to a known limiting distribution when the monotonicity condition in $H_{0,FRD}$ is true and diverges if the monotonicity condition is violated.

2.4 LFC and GMS Based Critical Values and Decision Rules

In this section, we introduce two simulated critical values for the proposed tests. The first is based on the least favorable condition (LFC), which is simple and popular but potentially conservative in moment inequality tests. The second is based on the generalized moment selection (GMS) method introduced by Andrews and Soares (2010) and Andrews and Shi (2013, 2015, 2017), which is often employed to improve the power of inequality tests over the LFC method.

We first introduce the simulated processes based on multiplier bootstrap. Let U_1, U_2,... be i.i.d. pseudo random variables with $E[U] = 0$, $E[U^2] = 1$ and $E[U^4] < \infty$ that are independent of the sample path. Define the simulated process

$$\widehat{\Phi}^u_{\mu,n}(\ell) = \sum_{i=1}^{n} U_i \cdot \hat{\phi}_{\mu,ni}(\ell). \tag{2.4}$$

Let P^u denote the multiplier probability measure. For significance level $\alpha < 1/2$, define the simulated critical value $\hat{c}_{n,FRD}(\alpha)$ based on the LFC as

$$\hat{c}^{\eta,LFC}_{n,FRD}(\alpha) = \sup \left\{ q \middle| P^u \Big(\sup_{\ell \in \mathcal{L}} \frac{\widehat{\Phi}^u_{\mu,n}(\ell)}{\hat{\sigma}_{\mu,\epsilon}(\ell)} \leq q \Big) \leq 1 - \alpha + \eta \right\} + \eta,$$

i.e. the $(1 - \alpha + \eta)$-th quantile of the simulated distribution of $\sup_{\ell \in \mathcal{L}} \frac{\widehat{\Phi}^u_{\mu,n}(\ell)}{\hat{\sigma}_{\mu,\epsilon}(\ell)}$ plus the small positive constant η. Define the decision rule based on the LFC critical value as "to reject $H_{0,FRD}$ if $\widehat{T}_{n,FRD} > \hat{c}^{\eta,LFC}_{n,FRD}(\alpha)$."

Let a_n and B_n be sequences of non-negative numbers, and η be an arbitrarily small positive number. Alternatively, we can use the GMS critical value $\hat{c}^{\eta,GMS}_{n,FRD}(\alpha)$ defined as

$$\hat{c}^{\eta,GMS}_{n,FRD}(\alpha) = \sup \left\{ q \middle| P^u \left(\sup_{\ell \in \mathcal{L}} \left(\frac{\widehat{\Phi}^u_{\mu,n}(\ell)}{\hat{\sigma}_{\mu,\epsilon}(\ell)} + \hat{\psi}_\mu(\ell) \right) \leq q \right) \leq 1 - \alpha + \eta \right\} + \eta,$$

$$\hat{\psi}_\mu(\ell) = -B_n \cdot 1 \left(\sqrt{nh} \cdot \frac{\hat{\mu}_n(\ell)}{\hat{\sigma}_{\mu,\epsilon}(\ell)} < -a_n \right).$$

Conditions of the a_n and B_n sequences are given in Assumption 3.4 in Section 3. Following Andrews and Shi (2013, 2015), we use $a_n = (0.3\ln(n))^{1/2}$, $B_n = (0.4\ln(n)/\ln\ln(n))^{1/2}$, and $\eta = 10^{-6}$. Define the decision rule based on the GMS critical value as "to reject $H_{0,FRD}$ if $\widehat{T}_{n,FRD} > \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)$."

3 Asymptotics of Proposed Tests

In this section, we study the asymptotic properties of the proposed tests based on both decision rules discussed in the previous section.

3.1 Regularity Conditions and Asymptotics of the Local Linear Estimators

Let $f_z(z)$ denote the probability density function (pdf) of Z, $f_{xz}(x,z)$ denote the pdf of X and Z. Let $\zeta_{P,+}(x,z) = E_P[Y|X = x, Z = z]$, $\sigma^2_{P,+}(x,z) = V_P(Y|X = x, Z = z)$ and $\varsigma_{P,+}(x,z) = E_P[T|X = x, Z = z]$ for $z \geq c$ and $\zeta_{P,-}(x,z) = E_P[Y|X = x, Z = z]$, $\sigma^2_{P,-}(x,z) = V_P(Y|X = x, Z = z)$ and $\varsigma_{P,-}(x,z) = E_P[T|X = x, Z = z]$ for $z < c$. Let $\mathcal{N}^+_{\delta,z}(c) = \{z : \ 0 \leq z - c \leq \delta\}$ be a neighborhood of Z from the cut-off value c to the right and $\mathcal{N}^-_{\delta,z}(c) = \{z : \ 0 < c - z \leq \delta\}$ be a neighborhood from c to the left. Let P_z denote the distribution of Z under P. Let $\mathcal{P}$ denote the collection of distributions P. We make the following assumptions.

Assumption 3.1 *Assume that for some $\delta > 0$ and for all $P \in \mathcal{P}$,*

(i) $\mathcal{X}_z = \mathcal{X}_c$ *for all* $z \in \mathcal{N}_{\delta,z}(c)$*;*

(ii) P_z *is the same and* $f_z(z)$ *is uniformly bounded away from zero and twice continuously differentiable in* z *on* $\mathcal{N}_{\delta,z}(c)$*;*

(iii) $f_{xz}(x,z)$ *is twice continuously differentiable in* z *on* $\mathcal{N}_{\delta,z}(c)$ *for all* $x \in \mathcal{X}_c$*, and* $\partial^2 f_{xz}(x,z)/\partial x \partial z$ *is uniformly bounded for all* $x \in \mathcal{X}_c$ *and* $z \in \mathcal{N}_{\delta,z}(c)$*;*

(iv) for all $x \in \mathcal{X}_c$*, both* $\zeta_{P,+}(x,z)$ *and* $\varsigma_{P,+}(x,z)$ *are twice continuously differentiable in* z *on* $\mathcal{N}^+_{\delta,z}(c)$*, and both* $\zeta_{P,-}(x,z)$ *and* $\varsigma_{P,-}(x,z)$ *are twice continuously differentiable in* z *on* $\mathcal{N}^-_{\delta,z}(c)$*;*

(v) $\partial\zeta_{P,+}(x,z)/\partial z$*,* $\partial^2\zeta_{P,+}(x,z)/\partial x\partial z$*,* $\partial\varsigma_{P,+}(x,z)/\partial z$*, and* $\partial^2\varsigma_{P,+}(x,z)/\partial x\partial z$ *are all uniformly bounded on* $\mathcal{X}_c\times\mathcal{N}^+_{\delta,z}(c)$*, and* $\partial z\zeta_{P,-}(x,z)/\partial z$*,* $\partial^2\zeta_{P,-}(x,z)/\partial x\partial z$*,* $\partial\varsigma_{P,-}(x,z)/\partial z$*, and* $\partial^2\varsigma_{P,-}(x,z)/\partial x\partial z$ *are all uniformly bounded on* $\mathcal{X}_c\times\mathcal{N}^-_{\delta,z}(c)$*;*

(vi) $\sigma^2_{P,+}(x,z)$ *is uniformly bounded and uniformly bounded away from zero on* $\mathcal{X}_c\times\mathcal{N}^+_{\delta,z}(c)$*;* $\sigma^2_{P,-}(x,z)$ *is uniformly bounded and uniformly bounded away from zero on* $\mathcal{X}_c\times\mathcal{N}^-_{\delta,z}(c)$*.*

(vii) $E_P[Y^4|Z=z]$ *is uniformly bounded for all* $z\in\mathcal{N}_{\delta,z}(c)$*;*

(viii) $E_P[T(1)-T(0)|Z=c]$ *is uniformly bounded away from zero.*

Assumption 3.1(i) is assumed for notational simplicity. We can allow $\mathcal{X}_z$ to depend on z and the theory will be the same, but it is more tedious in terms of notation. Assumption 3.1(ii)-(iv) are standard in nonparametric estimation. Assumption 3.1(v) is needed to show that the bias terms of the $\hat{\nu}(\ell)$ are asymptotically negligible uniformly over $\ell\in\mathcal{L}$ and over $P\in\mathcal{P}$. Assumption 3.1(vi) and (vii) are assumed so the covariance kernel estimator of the limiting process is uniformly consistent which is needed to show the validity of the multiplier bootstrap. Similar conditions are also assumed in Andrews and Shi (2015), Hsu (2016) and Hsu and Shen (2016). Assumption 3.1($viii$) is assumed such that the group of compliers which is the subpopulation of interest under fuzzy design is not of mass zero. Assumption 3.1(vi) and ($viii$) imply that $\sigma^2_{\mu,P}\left((\mathbf{1/2},\mathbf{0},\mathbf{0},1)\right)$ is bounded away from zero, which further implies that $\sigma^2_{\mu,\epsilon}(\ell)$ defined in Section 2 is bounded away from zero for all $\ell\in\mathcal{L}$.

Assumption 3.2 *Assume that*

(i) $K(\cdot)$ *is a non-negative symmetric bounded kernel with a compact support in* R*, and* $\int K(u)du=1$*;*

(ii) $h\to 0$*,* $nh\to\infty$ *and* $nh^5\to 0$ *as* $n\to\infty$*.*

Assumption 3.2(i) is a standard assumption on the kernel function. The triangular kernel ($K(u)=(1-|u|)\cdot 1(|u|\le 1)$), which is the most frequently used kernel function in RD estimation and testing, satisfies the conditions stated in the assumption. Assumption

3.2(ii) is the standard undersmoothing condition for local linear estimation. It helps eliminate the nuisance bias term and obtain centered asymptotic normality results of the local linear estimators.

Let $\ddot{m}(\ell) = (m^{(1)}(Y, W, S, \ell), m^{(2)}(Y, W, S, \ell), q^{(1)}(T, W, S, \ell), q^{(2)}(T, W, S, \ell))'$ be a 4×1 random vector. Let $h^{+}_{2,P}(\ell_1, \ell_2) = \lim_{z \searrow c} Cov_P(\ddot{m}(\ell_1), \ddot{m}(\ell_2)|Z = z)$, $h^{-}_{2,P}(\ell_1, \ell_2) = \lim_{z \nearrow c} Cov_P(\ddot{m}(\ell_1), \ddot{m}(\ell_2)|Z = z)$ be the left and right limit of its conditional variance-covariance matrix conditional on $Z = c$. For $j = 0, 1, 2$, let $\vartheta_j = \int_0^\infty u^j K(u)du$. Define $C_k = (\int_0^\infty (\vartheta_2 - u\vartheta_1)^2 K^2(u)du) / \big((\vartheta_2\vartheta_0 - \vartheta_1^2)^2 \cdot f_z(c)\big)$. Under Assumptions 3.1 and 3.2, we can show that

$$\sqrt{nh}\left(\left(\hat{m}^{(1)}_{n,+}(\cdot), \hat{m}^{(2)}_{n,+}(\cdot), \hat{q}^{(1)}_{n,+}(\cdot), \hat{q}^{(2)}_{n,+}(\cdot)\right)' - \left(m^{(1)}_{P,+}(\cdot), m^{(2)}_{P,+}(\cdot), q^{(1)}_{P,+}(\cdot), q^{(2)}_{P,+}(\cdot)\right)'\right) \Rightarrow \Phi_{C_k h^{+}_{2,P}}(\cdot),$$
$$\sqrt{nh}\left(\left(\hat{m}^{(1)}_{n,-}(\cdot), \hat{m}^{(2)}_{n,-}(\cdot), \hat{q}^{(1)}_{n,+}(\cdot), \hat{q}^{(2)}_{n,-}(\cdot)\right)' - \left(m^{(1)}_{P,+}(\cdot), m^{(2)}_{P,-}(\cdot), q^{(1)}_{P,-}(\cdot), q^{(2)}_{P,-}(\cdot)\right)'\right) \Rightarrow \Phi_{C_k h^{-}_{2,P}}(\cdot),$$

where $\Phi_{C_k h^{+}_{2,P}}$ and $\Phi_{C_k h^{-}_{2,P}}$ are independent mean zero Gaussian processes with covariance kernels $C_k h^{+}_{2,P}$ and $C_k h^{-}_{2,P}$, respectively. The weak convergence result stated above is a special case of Lemma B.1 in the appendix.

Further define that $h_{1,P}(\ell) = (-\varrho^{(2)}_P(\ell), \varrho^{(1)}_P(\ell), \rho^{(2)}_P(\ell), -\rho^{(1)}_P(\ell))$, then by the definitions of $\hat{\mu}_n$ and μ_P, we can show that

$$\sqrt{nh}(\hat{\mu}_n - \mu_P) \Rightarrow \Phi_{C_k h_{2,\mu,P}}$$

where $h_{2,\mu,P} = h_{1,P}(h^{+}_{2,P} + h^{-}_{2,P})h_{1,P}'$ is a mean zero Gaussian processes with covariance kernel $C_k h_{2,\mu,P}$. The weak convergence result stated above is a special case of Lemma B.2 in the appendix.

Assumption 3.3 *Let $\{U_i : 1 \leq i \leq n\}$ be a sequence of i.i.d. random variables that is independent of the sample path of $\{(Y_i, X_i, Z_i, T_i) : 1 \leq i \leq n\}$ such that $E[U_i] = 0$, $E[U_i^2] = 1$, and $E[|U_i|^4] < M$ for some $M > 0$.*

Assumption 3.3 is standard for the multiplier bootstrap and the bounded fourth moment of the simulated random variable is needed for the multiplier bootstrap of our nonparametric method. Under the assumptions, we can show that the simulated process $\widehat{\Phi}^u_{\mu,n}$ defined in Equation (2.4) converges in distribution to $\Phi_{C_k h_{2,\mu,P}}$, or the limiting process of the estimated moment functions $\hat{\mu}_n$, with probability approaching 1. See Lemma B.3 in the appendix for details.

Assumption 3.4 *Assume that*

(i) Let a_n *be a sequence of non-negative numbers satisfying* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} a_n/\sqrt{nh} = 0$*;*

(ii) Let B_n *be a sequence of non-negative numbers satisfying that* B_n *is non-decreasing,* $\lim_{n\to\infty} B_n = \infty$ *and* $\lim_{n\to\infty} B_n/a_n = 0$*.*

Assumption 3.4 follows from Andrews and Shi (2013, 2015) and is required for the validity of the GMS method.

3.2 Uniform Size Control and Consistency of the Proposed Tests

Define $\mathcal{H}_1 = \{h_{1,P}(\cdot) : P \in \mathcal{P}\}$, $\mathcal{H}_2^+ = \{h_{2,P}^+(\cdot,\cdot) : P \in \mathcal{P}\}$, and $\mathcal{H}_2^- = \{h_{2,P}^-(\cdot,\cdot) : P \in \mathcal{P}\}$. Let $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2^+ \times \mathcal{H}_2^-$. For any two $h = (h_1, h_2^+, h_2^-)$ and $\tilde{h} = (\tilde{h}_1, \tilde{h}_2^+, \tilde{h}_2^-)$ in the space of $\mathcal{H}$, define metric d as

$$
\begin{aligned}
d(h,\tilde{h}) &= \max\left\{d_1(h_1,\tilde{h}_1), d_2(h_2^+,\tilde{h}_2^+), d_2(h_2^-,\tilde{h}_2^-)\right\},\\
d_1(h_1,\tilde{h}_1) &= \sup_{\ell\in\mathcal{L}} \left\|h_1(\ell) - \tilde{h}_1(\ell)\right\|,\\
d_2(h_2,\tilde{h}_2) &= \sup_{\ell_1,\ell_2\in\mathcal{L}} \left\|h_2(\ell_1,\ell_2) - \tilde{h}_2(\ell_1,\ell_2)\right\|,
\end{aligned}
$$

where $\|\cdot\|$ denotes the Euclidean norm.

Assumption 3.5 *Let* $\mathcal{P}_0$ *be the subset of* $\mathcal{P}$ *that satisfies Assumption 3.1 such that the null hypothesis in (2.1) holds under* P *if* $P \in \mathcal{P}_0$*.*

Define $\mathcal{L}^o(P) = \{\ell : \mu_P(\ell) = 0\}$ which is the collection of indices satisfying $\mu_P(\ell) = 0$ under P. Then we have the following results of the proposed monotonicity test.

Theorem 3.1 *Suppose that Assumptions 3.1-3.5 hold. Then, for every compact subset of* $\mathcal{H}_{cpt}$ *of* $\mathcal{H}$*, we have*

(a) $\limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ h_P\in\mathcal{H}_{cpt}\}} P(\widehat{T}_{n,FRD} > \hat{c}_{n,FRD}^{\eta,LFC}(\alpha)) \leq \alpha$*;*

(b) $\limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ h_P\in\mathcal{H}_{cpt}\}} P(\widehat{T}_{n,FRD} > \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)) \leq \alpha$;

(c) if there exists $P_c^{LFC} \in \mathcal{P}_0$ *such that* $\mathcal{L}^o(P_c^{LFC}) = \mathcal{L}$ *and* $h_{2,\mu,P_c^{LFC}}$ *is not a zero function, then*

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ h_P\in\mathcal{H}_{cpt}\}} P(\widehat{T}_{n,FRD} > \hat{c}^{\eta,LFC}_{n,FRD}(\alpha)) = \alpha;$$

(d) if there exists $P_c \in \mathcal{P}_0$ *such that* $\mathcal{L}^o(P_c)$ *is not empty and* h_{2,μ,P_c} *restricted to* $\mathcal{L}^o(P_c)\times\mathcal{L}^o(P_c)$ *is not a zero function, then*

$$\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ h_P\in\mathcal{H}_{cpt}\}} P(\widehat{T}_{n,FRD} > \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)) = \alpha.$$

Parts (a) and (b) of Theorem 3.1 show that our tests based on both LFC and GMS critical values have uniform asymptotic size control over a compact subset of covariance kernels which is similar to Theorem 2(a) of Andrews and Shi (2013). Theorem 3.1(c) shows that our test based on the LFC critical value is at most infinitesimally conservative asymptotically when there exists at least one P_c^{LFC} satisfying the LFC condition: $\mu_P(\ell) = 0$ for all $\ell \in \mathcal{L}$. Theorem 3.1(d) shows that our test based on the GMS critical value is at most infinitesimally conservative asymptotically when there exists at least one P_c such that $\mathcal{L}^o(P_c)$ is not empty and h_{2,μ,P_c} restricted to $\mathcal{L}^o(P_c)\times\mathcal{L}^o(P_c)$ is not a zero function.

Last, we show that the proposed tests based on both decision rules are consistent, as is summarized in the following theorem.

Theorem 3.2 *Suppose that Assumptions 3.1-3.4 hold and that under* $P^* \in \mathcal{P}$, *there exist* $w_1^*, w_2^* \in \mathcal{W}$ *with* $w_2^* > w_1^*$ *and* $s^* \in \mathcal{S}$ *such that* $CLATE(w_2^*, s^*) < CLATE(w_1^*, s^*)$. *Then*

(a) $\lim_{n\to\infty} P^*(\widehat{T}_{n,FRD} > \hat{c}^{\eta,LFC}_{n,FRD}(\alpha)) = 1$.

(b) $\lim_{n\to\infty} P^*(\widehat{T}_{n,FRD} > \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)) = 1$.

4 Special Case: Sharp RD

When the treatment status is a deterministic function of the running variable such that $T = 1(Z \geq c)$, the RD model follows the sharp RD design. In this case, every individual is a complier, and the identification restrictions for the fuzzy RD case in Assumption 2.1 reduce to the following conditions.

Assumption 4.1 *For a running variable Z continuously distributed in $\mathcal{N}_{\delta,z}(c)$ for some $\delta > 0$,*

(i) $E_P[Y(t)|X = x, Z = z]$ *is continuous in x and z on $\mathcal{X}_c \times \mathcal{N}_{\delta,z}(c)$ for $t = 0, 1$.*

(ii) The distribution function of $X|Z = z$ is continuous in z on $\mathcal{N}_{\delta,z}(c)$.

Under Assumption 4.1, ATE and the conditional average treatment effect (CATE) conditional on X are identified as

$$ATE = \lim_{z \searrow c} E_P[Y|Z = z] - \lim_{z \nearrow c} E_P[Y|Z = z],$$
$$CATE(x) = \lim_{z \searrow c} E_P[Y|X = x, Z = z] - \lim_{z \nearrow c} E_P[Y|X = x, Z = z].$$

Recall that $X = (W, S)$. The test of interest then examines whether the $CATE(x)$ is monotonically increasing in w for all values of s. Mathematically, the null and alternative hypotheses could be written as

$H_{0,SRD} : CATE(x)$ is non-decreasing in w on $\mathcal{W}$ for all $s \in \mathcal{S}$;

$H_{1,SRD} :$ $H_{0,SRD}$ does not hold.

Under Assumption 4.1, it is easy to see that $CATE(x)$ is continuous in $x \in \mathcal{X}$. Similar to the discussion in Section 2.2, testing the null $H_{0,SRD}$ is equivalent to testing

$$H'_{0,SRD} : \nu_P(\ell) \equiv \rho_P^{(2)}(\ell) p_P^{(1)}(\ell) - \rho_P^{(1)}(\ell) p_P^{(2)}(\ell) \leq 0, \quad \text{for all } \ell \in \mathcal{L},$$

where $p_P^{(\kappa)} = E_P[g_{w_\kappa,\ell}(W) g_{s,\ell}(S)|Z = c]$ for $\kappa = 1, 2$, and $\mathcal{L}$ is defined in Lemma 2.2. First, we note that $p_P^{(\kappa)}$ is a special case of $\varrho_P^{(\kappa)}$ with $T = 1(Z \geq c)$. So the monotonicity test for $H'_{0,SRD}$ could be carried out using the same test statistic and decision rules for testing $H'_{0,FRD}$.

More efficiently, one could estimate $p_P^{(\kappa)}(\ell)$ by nonparametrically regressing $g_{w_\kappa,\ell}(W) g_{s,\ell}(S)$ on Z using data from both the left and the right of the cut-off c. Let $\hat{p}_n^{(\kappa)}(\ell)$ be the full-sample local linear estimator of $p_P^{(\kappa)}(\ell)$ which is the constant term of the following minimization problem:

$$\left(\hat{p}_n^{(\kappa)}(\ell), \hat{d}_n^{(\kappa)}(\ell)\right) = \arg\min_{a,b} \sum_{i=1}^{n} K\left(\frac{Z_i - c}{h}\right)\left[g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i) - a - b(Z_i - c)\right]^2.$$

For $l = 0, 1, 2$, define $S_{n,l} = \sum_{i=1}^{n} K(\frac{Z_i - c}{h})(Z_i - c)^l$, $\mathsf{w}_{ni} = \frac{K(\frac{Z_i - c}{h})[S_{n,2} - S_{n,1}(Z_i - c)]}{S_{0,2}S_{n,2} - S_{n,1}^2}$, and $p^{(\kappa)}(W_i, S_i, \ell) = g_{w_\kappa,\ell}(W_i) g_{s,\ell}(S_i)$. It then follows straightforwardly that $\hat{p}_n^{(\kappa)}(\ell) = \sum_{i=1}^{n} \mathsf{w}_{ni} \cdot p^{(\kappa)}(W_i, S_i, \ell)$ for both $\kappa = 1, 2$.

Let $\hat{\nu}_n(\ell) = \hat{\rho}_n^{(1)}(\ell)\hat{p}_n^{(2)}(\ell) - \hat{\rho}_n^{(2)}(\ell)\hat{p}_n^{(1)}(\ell)$ be the estimator of $\nu_P(\ell)$. Define $\phi_{p,ni}^{(\kappa)}(\ell) = w_{ni}(p^{(\kappa)}(W_i, S_i, \ell) - p_P^{(\kappa)}(\ell))$ for all $\ell \in \mathcal{L}$ and $\kappa = 1, 2$. Then the influence function representation of $\hat{\nu}_n(\ell)$ could be formulated as

$$\sqrt{nh}(\hat{\nu}_n(\ell) - \nu_P(\ell)) = \sum_{i=1}^{n} \phi_{\nu,ni}(\ell) + o_p(1),$$

where $\phi_{\nu,ni}(\ell)$ is defined similar to $\phi_{\mu,ni}(\ell)$ in Section 2 except that $\varrho_P^{(\kappa)}(\ell)$ and $\phi_{\varrho,ni}^{(\kappa)}(\ell)$ in $\phi_{\mu,ni}(\ell)$ are replaced by $p_P^{(\kappa)}(\ell)$ and $\phi_{p,ni}^{(\kappa)}(\ell)$ for both $\kappa = 1, 2$.

Let $\hat{\phi}_{\nu,ni}(\ell)$ be the estimated influence function. Similar to the fuzzy RD case, let the asymptotic variance of $\hat{\nu}_n(\ell)$ be estimated by $\hat{\sigma}_{\nu,n}^2(\ell) = \sum_{i=1}^{n} \hat{\phi}_{\nu,ni}(\ell)^2$ and the limiting process of $\sqrt{nh}(\hat{\nu}_n(\ell) - \nu_P(\ell))$ simulated by $\widehat{\Phi}_{\nu,n}^u(\ell) = \sum_{i=1}^{n} U_i \cdot \hat{\phi}_{\nu,ni}(\ell)$ with $U_1 ... U_n$ defined in Section 2.4. Let $\hat{\sigma}_{\nu,\epsilon}^2(\ell) = \max\left\{\hat{\sigma}_{\nu,n}^2(\ell), \epsilon \cdot \hat{\sigma}_{\nu,n}^2\left((\mathbf{1/2}, \mathbf{0}, \mathbf{0}, 1)\right)\right\}$. Then we can define the test statistic for the sharp RD case as

$$\widehat{T}_{n,SRD} = \sup_{\ell \in \mathcal{L}} \sqrt{nh} \frac{\hat{\nu}_n(\ell)}{\hat{\sigma}_{\nu,\epsilon}(\ell)},$$

and the LFC and GMS simulated critical values as

$$\hat{c}_{n,SRD}^{\eta,LFC}(\alpha) = \sup\left\{q \middle| P^u\Big(\sup_{\ell \in \mathcal{L}} \frac{\widehat{\Phi}_{\nu,n}^u(\ell)}{\hat{\sigma}_{\nu,\epsilon}(\ell)} \le q\Big) \le 1 - \alpha + \eta\right\} + \eta, \text{ and}$$

$$\hat{c}_{n,SRD}^{\eta,GMS}(\alpha) = \sup\left\{q \middle| P^u\left(\sup_{\ell \in \mathcal{L}}\left(\frac{\widehat{\Phi}_{\nu,n}^u(\ell)}{\hat{\sigma}_{\nu,\epsilon}(\ell)} + \hat{\psi}_\nu(\ell)\right) \le q\right) \le 1 - \alpha + \eta\right\} + \eta$$

respectively, with $\hat{\psi}_\nu(\ell) = -B_n \cdot 1\left(\sqrt{nh}\frac{\hat{\nu}_n(\ell)}{\hat{\sigma}_{\nu,\epsilon}(\ell)} < -a_n\right)$. If we reject the null hypothesis $H_{0,SRD}$ when $\widehat{T}_{n,SRD} > \hat{c}_{n,SRD}^{\eta,LFC}(\alpha)$ or when $\widehat{T}_{n,SRD} > \hat{c}_{n,SRD}^{\eta,GMS}(\alpha)$, the resulting tests are consistent and have uniform size control in the limit. The asymptotic properties are similar to those given in Section 3 for the fuzzy RD case and we omit the details for brevity.

5 Simulations

In this section, we investigate the small sample performance of the proposed tests. For all data generating processes (DGPs), the running variable Z, the additional control X, and the error term u in the outcome equation are generated following

$$Z \sim 2Beta(2,2) - 1;\ X \sim U[0,1];\ u \sim N(0,1).$$

The outcome Y and the treatment decision T are DGP specific. All DGPs are either estimated from the empirical example or modified from the data-driven DGPs to demonstrate specific properties of the proposed tests.

We use DGPs 1-3 to illustrate the small sample performance of the proposed monotonicity tests under the sharp RD design. DGPs 1-2 are estimated from the empirical dataset. DGP 3 is altered from DGP 3 to have a U-shaped CLATE to demonstrate the potential power gain of the GMS method. The DGPs are plotted in Figure 1 with detailed DGP generating procedures described in the footnote.

DGP 1: Sharp RD, Homogeneous Zero Effect

$$Y = \begin{cases} -0.373 + 0.545Z - 0.056Z^2 + 0.1u & \text{if } Z \geq 0 \\ -0.531 + 0.556Z + 0.192Z^2 + 0.1u & \text{if } Z > 0 \end{cases}$$

DGP 2: Sharp RD, Monotonically Increasing Treatment Effect

$$Y = \begin{cases} -0.755 - 0.254W + 0.742Z - 0.219WZ - 0.063Z^2 + 1.175W^2 + 0.1u & \text{if } Z \geq 0 \\ -0.607 - 0.220W + 0.386Z + 0.228WZ + 0.204Z^2 + 0.469W^2 + 0.1u & \text{if } Z < 0 \end{cases}.$$

DGP 3: Sharp RD, Inverse-U Shape Treatment Effect

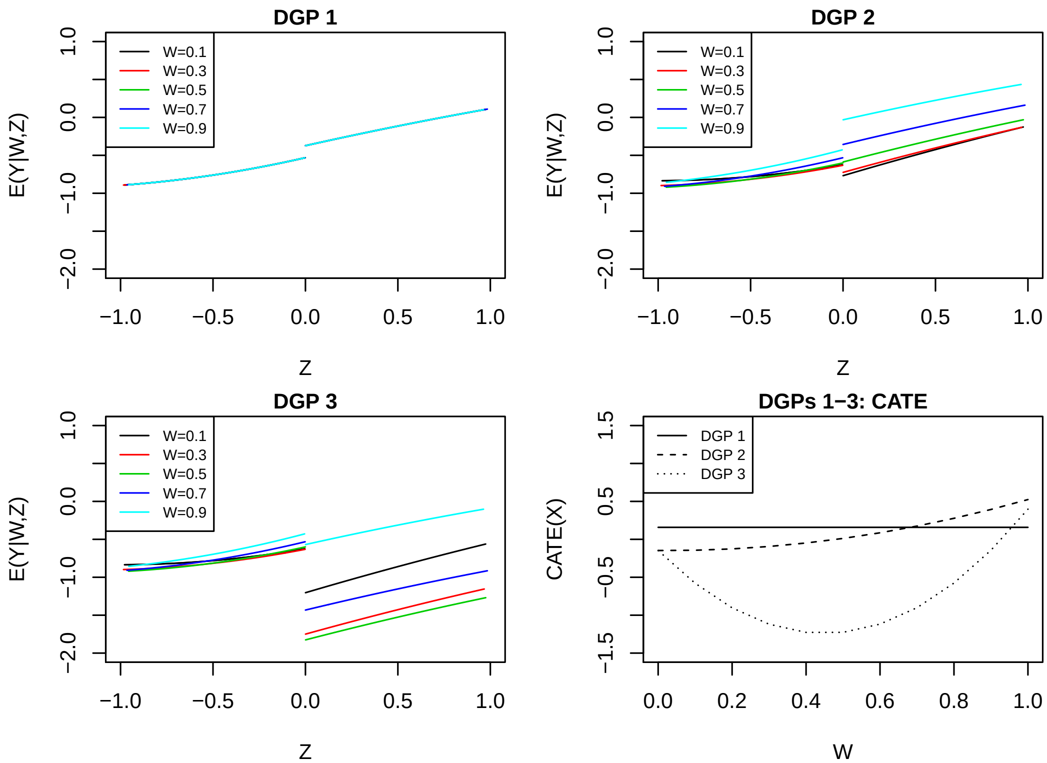
$$Y = \begin{cases} -0.755 - \mathbf{5.080}W + 0.742Z - 0.219WZ - 0.063Z^2 + \mathbf{5.875}W^2 + 0.1u & \text{if } Z \geq 0 \\ -0.607 - 0.220W + 0.386Z + 0.228WZ + 0.204Z^2 + 0.469W^2 + 0.1u & \text{if } Z < 0 \end{cases}.$$

The null hypotheses of interest are "$H_0 : CATE(x)$ is non-decreasing in x for all $x \in [0,1]$" and "$H_0 : CATE(x)$ is non-increasing in x for all $x \in [0,1]$". For each DGP, three different sample sizes $n = 2,000$, $n = 4,000$ and $n = 8,000$ are used. For each DGP and sample size combination, 2,000 simulation samples are drawn and the proposed monotonicity test is conducted in each Monte Carlo simulation. In each test, the bootstrap critical value is calculated using 1,000 bootstrap simulations. All tests carried out in this section use the triangular kernel and bandwidths selected according to the formula $h_{CCT} \times n^{1/5-1/k}$, where h_{CCT} is the robust bandwidth following Calonico et al. (2014) (CCT) and k is the under-smoothing parameter. In all simulation tables, we report results with $k = 4.25, 4.5$ and 4.75. The cubes defined in equation (2.3) have side-lengths $1/q$ for $q = 1, ..., Q$. We use benchmark $Q = 10$ which includes 165 combinations of overlapping $C_{w_1,10}$ and $C_{w_2,10}$ intervals. We also report robustness checks with $Q = 15$, which includes 560 combinations of overlapping $C_{w_1,15}$ and $C_{w_2,15}$ intervals. When $n = 2,000$ the bandwidth of DGPs 1-3 is around 0.18-0.25, which means that the expected effective sample size of the smallest $C_{w_1,15}$ and $C_{w_2,15}$ intervals ranges from 27 to 36 when $Q = 10$ and 18 to 24 when $Q = 15$ (for each local linear regression on one side of the RD cut-off).

Table 1 summarizes the rejection proportions of the proposed tests for the sharp RD case, with $p_P^{(\kappa)}(\ell)$ estimated using the full sample method as is discussed in the second half of Section 4. We see from the table that, regardless of whether the simulated critical value uses the LFC or the GMS method, the proposed monotonicity test controls size well in small samples and have power going to one as the sample size increases. The GMS method brings the size of the proposed test closer to the nominal level (5%) when the null of monotonic effect holds but not with equality (columns (1)-(3) and (7)-(9) in DGP 2) and increases the power when the null of monotonic effect is violated (DGP 3 and olumns (4)-(6) and (10)-(12) in DGP 2). The power gain is especially large for DGP 3. This is because, with the inverse U-Shape model in DGP3, the GMS method can get rid of the influence of about half of the hypercubes on critical value calculation when testing either direction of monotonicity. Last but not least, we notice that the simulation performance of the proposed test is not very sensitive to either the bandwidth choice or the choice of Q.

DGPs 4-6 illustrate the small sample performance of the proposed monotonicity tests under the fuzzy RD design. The outcome equation in these DGPs are the same as the

Figure 1: Data Generating Processes: GPPs 1-3



Note: The DGPs are estimated from the data of the empirical section. To obtain the models, we first rescale the running variable (i.e. transition score) in the data set to $[-1, 1]$ to match the support of the generated X variable. Then for the outcome equation in DGP 1, we regress the outcome (i.e. score in Baccalaureate exam) on the running variable and its second order polynomial term separately for the subsample to the left and the right of the cutoff value (i.e. 0). To get the outcome equation in DGP 2, we add the additional control of interest (i.e. average peer admission score), its second order polynomial term, and its interaction with the running variable to the set of regressors. To get the outcome equation in DGP 3 (bottom left graph), we take the model for DGP 2 and multiply the slope coefficient of the additional control variable by 20 and the slope coefficient of its second order polynomial term by 5.

Table 1: Small Sample Performance of Proposed Monotonicity Test Under Sharp RD

	LFC Critical Value						GMS Critical Value					
	H_0: Non-decreasing			H_0: Non-increasing			H_0: Non-decreasing			H_0: Non-increasing		
	c=4.25	4.5	4.75	4.25	4.5	4.75	4.25	4.5	4.75	4.25	4.5	4.75
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
						$Q = 10$						
	DGP 1: Homogeneous Zero Effect											
n=2000	0.019	0.020	0.022	0.017	0.020	0.020	0.024	0.026	0.026	0.019	0.023	0.022
n=4000	0.025	0.026	0.032	0.023	0.025	0.030	0.028	0.030	0.036	0.027	0.030	0.035
n=8000	0.042	0.042	0.043	0.039	0.039	0.041	0.048	0.045	0.048	0.044	0.044	0.044
	DGP 2: Monotonically Increasing Treatment Effect											
n=2000	0.005	0.005	0.006	0.329	0.373	0.418	0.009	0.008	0.009	0.334	0.378	0.429
n=4000	0.008	0.007	0.007	0.721	0.783	0.838	0.010	0.012	0.011	0.725	0.785	0.840
n=8000	0.006	0.005	0.005	0.970	0.981	0.990	0.011	0.012	0.013	0.970	0.981	0.990
	DGP 3: Inverse U-Shaped Treatment Effect											
n=2000	0.049	0.058	0.066	0.337	0.401	0.472	0.069	0.075	0.086	0.364	0.425	0.495
n=4000	0.180	0.205	0.236	0.833	0.895	0.927	0.214	0.244	0.277	0.844	0.900	0.929
n=8000	0.428	0.490	0.546	0.995	0.998	0.999	0.498	0.562	0.611	0.995	0.999	0.999
						$Q = 15$						
	DGP 1: Homogeneous Zero Effect											
n=2000	0.006	0.008	0.006	0.006	0.007	0.008	0.006	0.009	0.008	0.008	0.008	0.008
n=4000	0.010	0.012	0.013	0.010	0.012	0.014	0.012	0.014	0.014	0.010	0.012	0.015
n=8000	0.027	0.033	0.030	0.030	0.027	0.031	0.030	0.038	0.033	0.032	0.031	0.034
	DGP 2: Monotonically Increasing Treatment Effect											
n=2000	0.002	0.002	0.001	0.180	0.220	0.260	0.004	0.003	0.002	0.189	0.226	0.268
n=4000	0.004	0.002	0.004	0.556	0.636	0.710	0.006	0.006	0.006	0.564	0.640	0.712
n=8000	0.004	0.004	0.005	0.933	0.956	0.976	0.007	0.008	0.008	0.933	0.956	0.976
	DGP 3: Inverse U-Shaped Treatment Effect											
n=2000	0.017	0.022	0.030	0.170	0.219	0.272	0.023	0.029	0.036	0.184	0.234	0.296
n=4000	0.090	0.110	0.132	0.701	0.772	0.841	0.106	0.135	0.161	0.721	0.788	0.852
n=8000	0.320	0.390	0.453	0.984	0.992	0.997	0.374	0.449	0.500	0.986	0.992	0.997

Note: Reported are rejection proportions among 2,000 simulations, where all tests are carried out using the 5% significance level. For each test, the simulated critical value is calucalted with 1,000 bootstrap repetitions.

outcome equations in DGPs 1-3, respectively. The selection equation is modeled by

$$T = \begin{cases} 1(0.331 + 0.277Z + 0.049Z^2 + u > 0) & \text{if } Z \geq 0 \\ 0 & \text{if } Z > 0 \end{cases},$$

which is estimated from the data using a probit regression. Table 2 summarizes the rejection proportions of the proposed tests. We observe similar small sample performance as in the sharp RD case reproted in Table 1, although the tests generally have lower power under the fuzzy RD design due to the extra noise in the first stage.

6 Empirical Example: The Effect of Going to a Better High School

As is discussed in Pop-Eleches and Urquiola (2013), in Romania, a typical elementary school student takes a nationwide test in the last year of elementary school and applies to a list of high schools (and tracks). The admission decision is entirely dependent on the student's transition score, an average of the student's performance on the nationwide test and grade point average, as well as a student's preference for schools. A student is eligible to a high school if his/her transition score passes the school's cut-off.

Using an administrative dataset, Pop-Eleches and Urquiola (2013) find that attending a better school on average improves a student's performance on the Baccalaureate exam but does not significantly affect his/her probability of taking the exam. In this section, we apply the proposed monotonicity test to study how the effects of attending a more selective high school interact with peer quality of the school. As in Shen and Zhang (2015) and Hsu and Shen (2016), we focus on two-school towns because score cutoffs within a town are often quite close to each other and having more than one discontinuity point within the estimation window can introduce serious estimation bias.

In this application, the running variable (Z) is a student's standardized transition score subtracting individual school cut-off and the cut-off value (c) for having an offer from the more selective high school is zero for all students. The treatment variable (T) indicates whether a student attends the more selective high school in town. The outcome variable (Y) is a student's decision of whether to take the Baccalaureate exam

Table 2: Small Sample Performance of Proposed Monotonicity Test Under Fuzzy RD

	LFC Critical Value						GMS Critical Value					
	H_0: Non-decreasing			H_0: Non-increasing			H_0: Non-decreasing			H_0: Non-increasing		
	c=4.25	4.5	4.75	4.25	4.5	4.75	4.25	4.5	4.75	4.25	4.5	4.75
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
						$Q = 10$						
	DGP 4: Homogeneous Zero Effect											
n=2000	0.001	0.001	0.002	0.001	0.001	0.001	0.002	0.001	0.002	0.001	0.001	0.001
n=4000	0.002	0.002	0.002	0.004	0.005	0.005	0.003	0.003	0.004	0.004	0.005	0.006
n=8000	0.010	0.011	0.014	0.009	0.010	0.012	0.014	0.012	0.016	0.010	0.014	0.014
	DGP 5: Monotonically Increasing Treatment Effect											
n=2000	0.000	0.000	0.000	0.117	0.138	0.161	0.000	0.001	0.000	0.118	0.139	0.165
n=4000	0.000	0.000	0.001	0.404	0.468	0.546	0.001	0.001	0.001	0.407	0.472	0.550
n=8000	0.001	0.001	0.002	0.840	0.894	0.932	0.002	0.002	0.002	0.840	0.895	0.932
	DGP 6: Inverse U-Shaped Treatment Effect											
n=2000	0.021	0.024	0.030	0.127	0.170	0.214	0.028	0.032	0.038	0.139	0.185	0.234
n=4000	0.112	0.144	0.186	0.645	0.740	0.809	0.154	0.198	0.246	0.666	0.760	0.827
n=8000	0.479	0.568	0.658	0.994	0.998	1.000	0.574	0.664	0.742	0.994	1.000	1.000
						$Q = 15$						
	DGP 4: Homogeneous Zero Effect											
n=2000	0.001	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000
n=4000	0.001	0.000	0.000	0.001	0.002	0.002	0.0005	0.000	0.000	0.001	0.002	0.002
n=8000	0.008	0.009	0.011	0.008	0.009	0.009	0.011	0.012	0.014	0.010	0.013	0.013
	DGP 5: Monotonically Increasing Treatment Effect											
n=2000	0.000	0.000	0.000	0.056	0.072	0.088	0.000	0.000	0.000	0.058	0.076	0.090
n=4000	0.000	0.000	0.000	0.236	0.301	0.358	0.000	0.000	0.000	0.239	0.304	0.361
n=8000	0.001	0.001	0.001	0.635	0.689	0.762	0.002	0.001	0.002	0.640	0.699	0.768
	DGP 6: Inverse U-Shaped Treatment Effect											
n=2000	0.008	0.009	0.014	0.060	0.078	0.103	0.010	0.013	0.017	0.064	0.084	0.114
n=4000	0.046	0.066	0.086	0.448	0.552	0.646	0.066	0.085	0.112	0.466	0.574	0.666
n=8000	0.319	0.380	0.428	0.747	0.789	0.820	0.354	0.424	0.461	0.754	0.809	0.835

Note: Reported are rejection proportions among 2,000 simulations, where all tests are carried out using the 5% significance level. For each test, the simulated critical value is calucalted with 1,000 bootstrap repetitions.

(exam take) and his/her Baccalaureate exam grade (exam grade) conditional on taking the exam. The additional control variable (W) is the leave-one-out average transition score in the more selective high school in town, which is used to proxy the peer quality of the school. As is discussed in the simulation section, the proposed monotonicity test is carried out using the triangular kernel, the undersmoothed CCT bandwidth and the cubes defined in equation (2.3) with $Q = 50$ and 75. When $Q = 75$, for example, there is a total of 70,300 combinations of overlapping $C_{w_1,75}$ and $C_{w_2,75}$ intervals. The effective sample size of the smallest $C_{w_1,75}$ and $C_{w_2,75}$ intervals ranges from 50 to 62 (for each local linear regression on one side of the RD cut-off). Notice that in the simulation section the effective sample size of smallest cubes is kept around 20. For the empirical sample, due to its very large sample size, this rule-of-thumb will result in a Q value as large as around 200 and a total of 1,333,300 combinations of overlapping $C_{w_1,200}$ and $C_{w_2,200}$ intervals, which is computationally infeasible. Instead, we report results with both $Q = 50$ and 75. We find our empirical results insensitive to the Q choice.

Table 3 reports the results of the monotonicity tests. First we discuss the effect of attending a more selective high school on a student's probability of taking the Baccalaureate exam. As we see from the table, regardless of choice in the under-smoothing parameter, the number of cubes, or whether to use the LFC or GMS critical value, we fail to reject the null of monotonically non-decreasing effect with very high p-values and reject the null of monotonically non-increasing effect with p-values ranging between 1-3%. These testing results suggest that the effect on exam taking rate monotonically increases with peer quality of the more selective school. The finding is in line with the results of uniform sign tests conducted in Hsu and Shen (2016), which find that the effect of attending a more selective high school on the Baccalaureate exam taking rate is positive for some subpopulation of schools and negative for the others. Together, we can conclude that the insignificant mean effect found in Pop-Eleches and Urquiola (2013) is due to cancelation of positive and negative treatment effects among schools with different peer qualities.

In contrast, the effect on the exam grade outcome is likely homogeneous among schools with different peer quality as we fail to reject both null of monotonically non-decreasing and monotonically non-increasing effects at the 10% significance level. Note that the p-values associated with the GMS method are always the same or smaller than those

Table 3: P-values of Monotonicity Tests

	LFC			GMS		
	c=4.5	c=4.75	c=5	c=4.5	c=4.75	c=5
			$Q = 50$			
H_0: the effect is non-decreasing						
First Stage	0.001	0.000	0.000	0.001	0.000	0.000
Exam Take	0.801	0.794	0.747	0.776	0.774	0.723
Exam Grade	0.360	0.335	0.353	0.347	0.322	0.336
H_0: the effect is non-increasing						
First Stage	0.032	0.024	0.016	0.025	0.019	0.012
Exam Take	0.027	0.016	0.009	0.027	0.016	0.009
Exam Grade	0.196	0.144	0.140	0.196	0.144	0.139
			$Q = 75$			
H_0: the effect is non-decreasing						
First Stage	0.001	0.000	0.000	0.001	0.000	0.000
Exam Take	0.807	0.802	0.757	0.786	0.787	0.734
Exam Grade	0.374	0.349	0.345	0.362	0.333	0.326
H_0: the effect is non-increasing						
First Stage	0.032	0.024	0.016	0.025	0.019	0.012
Exam Take	0.027	0.016	0.009	0.027	0.016	0.009
Exam Grade	0.219	0.163	0.156	0.215	0.159	0.152

Notes: Data are from Pop-Eleches and Urquiola (2013). Nonparametric local linear estimations are conducted using the triangular kernel, the undersmoothed CCT bandwidth defined in the simulation section, and the cubes defined in equation (2.3). All simulated critical values are calculated with 1,000 bootstrap repetitions.

associated with the LFC method. This is because the GMS method potentially improves the power of the proposed monotonicity test, as is discussed in the theoretical section of the paper.

We also conduct in Table 3 monotonicity tests for the first stage enrollment decision. The tests are carried out using the sharp RD test discussed in Section 4. We reject both null of monotonically non-decreasing and monotonically non-increasing first stage effects with very small p-values. The testing results suggest that the first-stage enrollment decision is strongly heterogenous, but its relationship with peer quality in the more selective high school is not simple and not monotonic.

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APPENDIX

A Proofs of Lemmas in Section 2

This section proves the Lemmas in Section 2 that illustrate the equivalence between the original null hypothesis in (2.1) to the transformed null hypothesis in (2.2).

Proof of Lemma 2.1:

For notational simplicity and without loss of generality, we prove the lemma for the case with $d_w = d_s = 1$. Proof for the higher dimension case follows with the same idea but require more complicated notations.

First, we prove that (i) implies (ii). For any $w_1 = w_2$, the inequality in (ii) holds trivially with equality. For any $w_1 > w_2$ and any $q = 1, 2, 3, \ldots$, such that $(q+1) \cdot w_1, (q+1) \cdot w_2 \in \{0, 1, 2, \cdots, q\}$, we have $w_1 \geq w_2 + 1/(q+1)$. By (i), this implies that $\lambda(w, s) \geq \lambda(w', s)$ for all $w \in [w_1, w_1 + 1/(q+1)]$ and $w' \in [w_2, w_2 + 1/(q+1)]$ and $s \in \mathcal{S}$. Therefore, the weighted average of $\lambda(w, s)$ over $w \in [w_1, w_1 + /(q+1)]$ and $s \in C_{s,q}$ has to be greater or equal to that of $\lambda(w', s)$ over $w' \in [w_2, w_2 + 1/(q+1)]$ and $s \in C_{s,q}$. Equivalently,

$$\frac{\int_{C_{w_1,q} \times C_{s,q}} \lambda(\tilde{w}, \tilde{s}) \cdot h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}}{\int_{C_{w_1,q} \times C_{s,q}} h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}} \geq \frac{\int_{C_{w_2,q} \times C_{s,q}} \lambda(\tilde{w}, \tilde{s}) \cdot h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}}{\int_{C_{w_2,q} \times C_{s,q}} h(\tilde{w}, \tilde{s}) d\tilde{w} d\tilde{s}}.$$

We prove the inequality in (ii).

Second, we prove that (ii) implies (i) by contradiction. Suppose that $\lambda(w, s') < \lambda(w', s')$ for some s' and $w \geq w'$. By continuity of $\lambda(w, s)$, there exist $w_u > w_l > w'_u > w'_l$ and $s_u > s_l$ so that $\lambda(w, s') < \lambda(w', s')$ for all $w \in [w_l, w_u]$, $w' \in [w'_l, w'_u]$ and $s' \in [s_l, s_u]$. In turn, we can find a q large enough such that for some w_1, w_2, and s, $(q+1) \cdot w_1, (q+1) \cdot w_2 \in \{0, 1, 2, \cdots, q\}$, and $q \cdot s \in \{0, 1, 2, \cdots, q-1\}$ so that $[w_1, w_1 + 1/(q+1)] \subseteq [w_l, w_u]$, $[w_2, w_2 + 1/(q+1)] \subseteq [w'_l, w'_u]$ and $[s, s + 1/q] \subseteq [s_l, s_u]$. Then the weighted average of $\lambda(w, s)$ over $w \in [w_1, w_1 + 1/(q+1)]$ and $s \in [s, s + 1/q]$ has to be strictly less than that of $\lambda(w', s)$ over $w' \in [w_2, w_2 + 1/(q+1)]$ and $s \in [s, s + 1/q]$.

That is,

$$\frac{\int_{C_{w_1,q}\times C_{s,q}} \lambda(\tilde{w},\tilde{s})\cdot h(\tilde{w},\tilde{s})d\tilde{w}d\tilde{s}}{\int_{C_{w_1,q}\times C_{s,q}} h(\tilde{w},\tilde{s})d\tilde{w}d\tilde{s}} < \frac{\int_{C_{w_1,q}\times C_{s,q}} \lambda(\tilde{w},\tilde{s})\cdot h(\tilde{w},\tilde{s})d\tilde{w}d\tilde{s}}{\int_{C_{w_1,q}\times C_{s,q}} h(\tilde{w},\tilde{s})d\tilde{w}d\tilde{s}}.$$

This completes our proof. □

Proof of Lemma 2.2:

To prove the lemma, we first apply the results of Lemma 2.1. Let $\lambda(w,s) = CLATE(w,s) = E_P[(Y(1)-Y(0)(T(1)-T(0))|W=w,S=s,Z=c]/E_P[T(1)-T(0)|W=w,S=s,Z=c]$ and $h(w,s) = E_P[T(1)-T(0)|W=w,S=s,Z=c]\cdot f_{W,S|Z=c}(w,s)$. Then, for both $\kappa = 1,2$,

$$\begin{aligned}
&\int_{C_{w_\kappa,q}\times C_{s,q}} \lambda(w,s)\cdot h(w,s)dwds\\
=&\int_{C_{w_\kappa,q}\times C_{s,q}} \big(\lim_{z\searrow c} E_P[Y|W=w,S=s,Z=z]\\
&\qquad\qquad - \lim_{z\nearrow c} E_P[Y|W=w,S=s,Z=z]\big)\cdot f_{W,S|Z=c}(w,s)dwds\\
=&\lim_{z\searrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)Y|Z=z] - \lim_{z\nearrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)Y|Z=z],\\
\equiv&\rho_P^{(\kappa)}(\ell),
\end{aligned}$$

where the second equality holds by the continuity of the conditional means and by law of iterated expectations. Similarly,

$$\begin{aligned}
&\int_{C_{w_\kappa,q}\times C_{s,q}} h(w,s)dwds\\
=&\int_{C_{w_\kappa,q}\times C_{s,q}} \big(\lim_{z\searrow c} E_P[T|W=w,S=s,Z=z]\\
&\qquad\qquad - \lim_{z\nearrow c} E_P[T|W=w,S=s,Z=z]\big)\cdot f_{W,S|Z=c}(w)dwds\\
=&\lim_{z\searrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)T|Z=z] - \lim_{z\nearrow c} E_P[g_{w_\kappa,\ell}(W)g_{s,\ell}(S)T|Z=z]\\
\equiv&\varrho_P^{(\kappa)}(\ell).
\end{aligned}$$

So testing the null hypothesis $H_{0,FRD}$ is equivalent to testing

$$\rho_P^{(2)}(\ell)\varrho_P^{(1)}(\ell) - \rho_P^{(1)}(\ell)\varrho_P^{(2)}(\ell) \leq 0$$

for all $q = 1,2,3,\cdots$, and $w_1 \geq w_2$ such that $(q+1)\cdot(w_1,w_2) \in \{0,1,2,\cdots,q\}^{2d_w}$, and $q\cdot s \in \{0,1,2,\cdots,q-1\}^{d_s}$.

B Proofs of Theorems in Section 3

To prove the theorems for the asymptotic properties of the proposed tests, we first give auxiliary lemmas that will be used in the proof of the main results. Then we give the proofs of the main theorems in Section 3.

B.1 Auxiliary lemmas

Let E_Z denote the expectation conditional on sample path $\{Z_1, Z_2, \ldots\}$.

Lemma B.1 *Given Assumptions 3.1 and 3.2 hold, for any subsequence of k_n of n such that for $\lim_{n\to\infty} d_2(h^{+}_{2,P_{k_n}}, h^{*+}_2) = 0$ for some $h^{*+}_2 \in \mathcal{H}^{+}_2$ and $\lim_{n\to\infty} d_2(h^{-}_{2,P_{k_n}}, h^{*-}_2) = 0$ for some $h^{*-}_2 \in \mathcal{H}^{-}_2$, we have*

$$\sqrt{k_n h}\begin{pmatrix} \hat{m}^{(1)}_{k_n,+}(\cdot) - m^{(1)}_{P_{k_n},+}(\cdot) \\ \hat{m}^{(2)}_{k_n,+}(\cdot) - m^{(2)}_{P_{k_n},+}(\cdot) \\ \hat{q}^{(1)}_{k_n,+}(\cdot) - q^{(1)}_{P_{k_n},+}(\cdot) \\ \hat{q}^{(2)}_{k_n,+}(\cdot) - q^{(2)}_{P_{k_n},+}(\cdot) \end{pmatrix} \Rightarrow \Phi_{C_k h^{*+}_2}(\cdot)\ , \quad \sqrt{k_n h}\begin{pmatrix} \hat{m}^{(1)}_{k_n,-}(\cdot) - m^{(1)}_{P_{k_n},-}(\cdot) \\ \hat{m}^{(2)}_{k_n,-}(\cdot) - m^{(2)}_{P_{k_n},-}(\cdot) \\ \hat{q}^{(1)}_{k_n,-}(\cdot) - q^{(1)}_{P_{k_n},-}(\cdot) \\ \hat{q}^{(2)}_{k_n,-}(\cdot) - q^{(2)}_{P_{k_n},-}(\cdot) \end{pmatrix} \Rightarrow \Phi_{C_k h^{*-}_2}(\cdot)\ ,$$

where $\Phi_{C_k h^{*+}_2}$ and $\Phi_{C_k h^{*-}_2}$ are independent mean zero Gaussian processes with covariance kernel $C_k h^{*+}_2$ and $C_k h^{*-}_2$.

Proof of Lemma B.1:

Here we prove the first weak convergence result. The second one follows by essentially the same proof but with the $+$ notation replaced by $-$. The resulting two limiting Gaussian processes are independent because the local linear estimators involved in the two convergence results using different subsamples and the data are independent.

By Cramér-Wold Theorem, it is sufficient to show the $m^{(1)}_{P_{k_n},+}(\ell)$ case. Note that

$$
\begin{aligned}
&\sqrt{k_n h}(\hat{m}^{(1)}_{k_n,+}(\ell) - m^{(1)}_{P_{k_n},+}(\ell)) \\
=&\sum_{i=1}^{k_n} \sqrt{k_n h}(\mathsf{w}^{+}_{k_n i}(m^{(1)}(Y_i, W_i, S_i, \ell) - m^{(1)}_{P_{k_n},+}(\ell))) \\
=&\sum_{i=1}^{k_n} \sqrt{k_n h}(\mathsf{w}^{+}_{k_n i}(m^{(1)}(Y_i, W_i, S_i, \ell) - E_Z[m^{(1)}(Y_i, W_i, S_i, \ell)])) \\
&+ \sum_{i=1}^{k_n} \sqrt{k_n h}(\mathsf{w}^{+}_{k_n i}(E_Z[m^{(1)}(Y_i, W_i, S_i, \ell)] - m^{(1)}_{P_{k_n},+}(\ell))).
\end{aligned}
$$

We first consider the second term which is the bias term. By Theorem 4 of Fan and Gijbels (1992), we know that

$$
\sum_{i=1}^{k_n} \sqrt{k_n h}(\mathsf{w}^{+}_{k_n i}(E_Z[m^{(1)}(Y_i, W_i, S_i, \ell)] - m^{(1)}_{P_{k_n},+}(\ell))) = O_p(\sqrt{k_n h^5}) = o_p(1).
$$

Note that the first equality holds because the magnitude of the bias is proportional to the second derivative of $m^{(1)}_{P_{k_n},+}(\ell)$ with respect to z. By Assumption 3.1, we know that for all P_{k_n}, $\partial^2 \zeta_{P_{k_n},+}(x,z)/\partial z \partial z$ is uniformly bounded on $x \in \mathcal{X}_c$ and $z \in \mathcal{N}^{+}_{\delta,z}(c)$. Since $m^{(1)}_{P_{k_n},+}(\ell) = \lim_{z \searrow c} E_{P_{k_n}}[g_{w_1,\ell}(W) g_{s,\ell}(S) Y | Z = z]$, $m^{(1)}_{P_{k_n},+}(\ell)$ is uniformly bounded as well. Then given the additional assumption that $k_n h^5 \to 0$, we know that the above $o_p(1)$ result holds uniformly over $\ell \in \mathcal{L}$.

Therefore, uniformly over $\ell \in \mathcal{L}$, we have

$$
\begin{aligned}
&\sqrt{k_n h}(\hat{m}^{(1)}_{k_n,+}(\ell) - m^{(1)}_{P_{k_n},+}(\ell)) \\
=&\sum_{i=1}^{k_n} \sqrt{k_n h}(\mathsf{w}^{+}_{k_n i}(m^{(1)}(Y_i, W_i, S_i, \ell) - E_Z[m^{(1)}(Y_i, W_i, S_i, \ell)])) + o_p(1).
\end{aligned}
$$

It is then easy to show that $\{(m^{(1)}(Y_i, W_i, S_i, \ell) :\ 1 \le i \le k_n,\ n \ge 1\}$ satisfies the manageability condition in the functional central limit theorem (FCLT), or Theorem 10.6 of Pollard (1990). The arguments are similar to those in the proof of Lemma 3.2 of Hsu and Shen (2016) and hold along the sequence $\{P_{k_n} :\ n \ge 1\}$. □

Lemma B.2 *Suppose that Assumptions 3.1 and 3.2 hold. For any subsequence of k_n of*

n such that for $\lim_{n\to\infty} d(h_{P_{k_n}}, h^*) = 0$ for some $h^* \in \mathcal{H}$, we have

$$\sup_{\ell\in\mathcal{L}} \Big| \sqrt{k_n h}(\hat{\mu}_{k_n}(\ell) - \mu_{P_{k_n}}(\ell)) - \sum_{i=1}^{k_n} \phi_{\mu,k_n i}(\ell)| = o_p(1), \ \textit{and}$$
$$\sqrt{k_n h}(\hat{\mu}_{k_n}(\cdot) - \mu_{P_{k_n}}(\cdot)) \Rightarrow \Phi_{C_k h^*_{2,\mu}}$$

where $h^*_{2,\mu} = h^*_1(h^{*+}_2 + h^{*-}_2)h^{*\prime}_1$.

Proof of Lemma B.2: First, note that Lemma B.1 implies that for $\kappa = 1, 2$,

$$\sup_{\ell\in\mathcal{L}} |\hat{\rho}^{(\kappa)}_{k_n}(\ell) - \rho^{(\kappa)}_{P_{k_n}}(\ell)| = O_p((k_n h)^{-1/2}),$$
$$\sup_{\ell\in\mathcal{L}} |\hat{\varrho}^{(\kappa)}_{k_n}(\ell) - \varrho^{(\kappa)}_{P_{k_n}}(\ell)| = O_p((k_n h)^{-1/2}). \tag{B.1}$$

Next, note that

$$\begin{aligned}
&\sqrt{k_n h}(\hat{\rho}^{(2)}_{k_n}(\ell)\hat{\varrho}^{(1)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\varrho^{(1)}_{P_{k_n}}(\ell)) \\
=&\hat{\varrho}^{(1)}_{k_n}(\ell)\sqrt{k_n h}\big(\hat{\rho}^{(2)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\big) - \rho^{(2)}_{P_{k_n}}(\ell)\sqrt{k_n h}\big(\hat{\varrho}^{(1)}_{k_n}(\ell) - \varrho^{(1)}_{P_{k_n}}(\ell)\big) \\
=&\varrho^{(1)}_{P_{k_n}}(\ell)\sqrt{k_n h}\big(\hat{\rho}^{(2)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\big) - \rho^{(2)}_{P_{k_n}}(\ell)\sqrt{k_n h}\big(\hat{\varrho}^{(1)}_{k_n}(\ell) - \varrho^{(1)}_{P_{k_n}}(\ell))\big) \\
&+ \sqrt{k_n h}\big(\hat{\rho}^{(2)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\big)\big(\hat{\varrho}^{(1)}_{k_n}(\ell) - \varrho^{(1)}_{P_{k_n}}(\ell))\big) \\
=&\varrho^{(1)}_{P_{k_n}}(\ell)\sqrt{k_n h}\big(\hat{\rho}^{(2)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\big) - \rho^{(2)}_{P_{k_n}}(\ell)\sqrt{k_n h}\big(\hat{\varrho}^{(1)}_{k_n}(\ell) - \varrho^{(1)}_{P_{k_n}}(\ell))\big) + o_p(1)
\end{aligned}$$

where the $o_p(1)$ result holds uniformly over $\ell \in \mathcal{L}$ due to Equation (B.1). Further, since

$$\sqrt{k_n h}(\hat{\rho}^{(\kappa)}_{k_n}(\ell) - \rho^{(\kappa)}_{P_{k_n}}(\ell)) = \sum_{i=1}^{k_n} \phi^{(\kappa)}_{\rho,k_n i}(\ell), \qquad \sqrt{k_n h}(\hat{\varrho}^{(\kappa)}_{k_n}(\ell) - \varrho^{(\kappa)}_{P_{k_n}}(\ell)) = \sum_{i=1}^{k_n} \phi^{(\kappa)}_{\varrho,k_n i}(\ell),$$

for all $\ell \in \mathcal{L}$, we have that

$$\begin{aligned}
&\sqrt{k_n h}(\hat{\rho}^{(2)}_{k_n}(\ell)\hat{\varrho}^{(1)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\varrho^{(1)}_{P_{k_n}}(\ell)) \\
=&\sum_{i=1}^{k_n} \varrho^{(1)}_{P_{k_n}}(\ell)\phi^{(2)}_{\rho,k_n i}(\ell) - \sum_{i=1}^{k_n} \rho^{(2)}_{P_{k_n}}(\ell)\phi^{(1)}_{\varrho,k_n i}(\ell) + o_p(1).
\end{aligned}$$

and the $o_p(1)$ result holds uniformly over $\ell \in \mathcal{L}$.

Similarly, we can write

$$\begin{aligned}
&\sqrt{k_n h}(\hat{\rho}^{(1)}_{k_n}(\ell)\hat{\varrho}^{(2)}_{k_n}(\ell) - \rho^{(1)}_{P_{k_n}}(\ell)\varrho^{(2)}_{P_{k_n}}(\ell)) \\
=&\sum_{i=1}^{k_n} \varrho^{(2)}_{P_{k_n}}(\ell)\phi^{(1)}_{\rho,k_n i}(\ell) - \sum_{i=1}^{k_n} \rho^{(1)}_{P_{k_n}}(\ell)\phi^{(2)}_{\varrho,k_n i}(\ell) + o_p(1).
\end{aligned}$$

Finally, we have

$$
\begin{aligned}
&\sqrt{k_n h}(\hat{\mu}_{k_n}(\ell) - \mu_{P_{k_n}}(\ell)) \\
=&\sqrt{k_n h}(\hat{\rho}^{(2)}_{k_n}(\ell)\hat{\varrho}^{(1)}_{k_n}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\varrho^{(1)}_{P_{k_n}}(\ell) - \hat{\rho}^{(1)}_{k_n}(\ell)\hat{\varrho}^{(2)}_{k_n}(\ell) + \rho^{(1)}_{P_{k_n}}(\ell)\varrho^{(2)}_{P_{k_n}}(\ell)) \\
=&\sum_{i=1}^{k_n} \varrho^{(1)}_{P_{k_n}}(\ell)\phi^{(2)}_{\rho,k_n i}(\ell) - \rho^{(2)}_{P_{k_n}}(\ell)\phi^{(1)}_{\varrho,k_n i}(\ell) - \varrho^{(2)}_{P_{k_n}}(\ell)\phi^{(1)}_{\rho,k_n i}(\ell) + \rho^{(1)}_{P_{k_n}}(\ell)\phi^{(2)}_{\varrho,k_n i}(\ell) + o_p(1) \\
=&\sum_{i=1}^{k_n} \phi_{\mu,k_n i}(\ell) + o_p(1).
\end{aligned}
$$

with the $o_p(1)$ result holding uniformly over $\ell \in \mathcal{L}$.

The above equation shows the first part of Lemma B.2. To show the second part, we will apply the Theorem 10.6 of Pollard (1990). Define our triangular array as $\{\phi_{\mu,k_n i}(\ell):\ \ \ell \in \mathcal{L},\ i \le k_n,\ 1 \le n\}$. Note the by the same argument as in Lemma A1 of Hsu et al. (2017), we can show that the triangular is manageable so the part (i) of Theorem 10.2 of Pollard (1990) holds. We can apply similar arguments of Lemma 3.2 of Hsu and Shen (2016) to show that Parts (ii)-(v) hold too. These would complete our proof and we omit the details for brevity. □

Lemma B.3 *Assume that Assumptions 3.1, 3.2, and 3.3 hold. For any subsequence of k_n of n such that for $\lim_{n\to\infty} d(h_{P_{k_n}}, h^*) = 0$ for some $h^* \in \mathcal{H}$, we have the simulated process $\widehat{\Phi}^u_{\mu,n}(\cdot) \Rightarrow \Phi_{C_k h^*_{2,\mu}}(\cdot)$ conditional on sample path with probability approaching 1.*

Proof of Lemma B.3: Recall that $\widehat{\Phi}^u_{\mu,k_n}(\cdot) = \sum_{i=1}^{k_n} U_i \cdot \hat{\phi}_{\mu,k_n i}(\cdot)$. It is straightforward to see that $\{U_i \cdot \hat{\phi}_{\mu,k_n i}(\ell):\ \ell \in \mathcal{L},\ i \le k_n,\ 1 \le n\}$ is manageable. Define $\ddot{h}_{2,k_n,\mu}(\ell_1,\ell_2) = \sum_{i=1}^{k_n} \hat{\phi}_{\mu,k_n i}(\ell_1)\hat{\phi}_{\mu,k_n i}(\ell_2)$. We know that $\sup_{\ell_1,\ell_2\in\mathcal{L}} |\ddot{h}_{2,k_n,\mu}(\ell_1,\ell_2) - C_k h^*_{2,\mu}(\ell_1,\ell_2)| \xrightarrow{p} 0$. The rest of the proof is similar to that for Lemma 3.3 of Hsu and Shen (2016) and we omit the details. □

Lemma B.4 *Assume that Assumptions 3.1 and 3.2 hold. For any subsequence of k_n of n such that for $\lim_{n\to\infty} d(h_{P_{k_n}}, h^*) = 0$ for some $h^* \in \mathcal{H}$, then $\sup_{\ell\in\mathcal{L}} |\hat{\sigma}_{\mu,\epsilon}(\ell)^{-1} - \sigma^*_{\mu,\epsilon}(\ell)^{-1}| \xrightarrow{p} 0$, where $\sigma^*_{\mu,\epsilon}(\ell) = \max\{\sigma^*_\mu(\ell), \epsilon\cdot\sigma^*_\mu((\boldsymbol{1/2}, \boldsymbol{0}, \boldsymbol{0}, 1))\}$ and $\sigma^*_\mu(\ell) = \big(C_k h^*_{2,\mu}(\ell,\ell)\big)^{1/2}$.*

Proof of Lemma B.4: As in the proof of Lemma B.3, we can show that $\sup_{\ell\in\mathcal{L}} |\hat{\sigma}_{\mu,k_n} - \sigma^*_\mu(\ell)| \xrightarrow{p} 0$. Next, by the fact that the maximum operator is a continuous functional, we have $\sup_{\ell\in\mathcal{L}} |\hat{\sigma}_{\mu,\epsilon}(\ell) - \sigma^*_{\mu,\epsilon}(\ell)| \xrightarrow{p} 0$. Note that $\sigma^*_{\mu,\epsilon}(\ell)$ is bounded away from zero because we can show that $\sigma^*_{\mu}((\mathbf{1/2}, \mathbf{0}, \mathbf{0}, 1))$ is bounded away from zero, so it follows that $\sup_{\ell\in\mathcal{L}} |\hat{\sigma}_{\mu,\epsilon}(\ell)^{-1} - \sigma^*_{\mu,\epsilon}(\ell)^{-1}| \xrightarrow{p} 0$. This completes the proof of Lemma B.4. $\square$

B.2 Proofs of Theorems

Proof of Theorem 3.1: Note that by construction, $\hat{c}^{\eta,LFC}_{n,FRD}(\alpha) \geq \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)$, so the size of the test based on the LFC critical value is always smaller than that based on $\hat{c}^{\eta,GMS}_{n,FRD}(\alpha)$. Therefore, it is sufficient to show that the test based on $\hat{c}^{\eta,GMS}_{n,FRD}(\alpha)$ has uniform size control. To show this, we can apply the same arguments as in the proof of Theorem 4.1 of Hsu et al. (2017) given Lemmas B.2, B.3 and B.4 and we omit the details.

To show part (d) of the theorem, note that if there exists $P_c \in \mathcal{P}_0$ such that $\mathcal{L}^o(P_c)$ is not empty and h_{2,μ,P_c} restricted to $\mathcal{L}^o(P_c) \times \mathcal{L}^o(P_c)$ is not a zero function, then by the same proof based on the pointwise asymptotics as in Lemma 1 of Donald and Hsu (2016) and by Tsirel'Son (1976), we have that under P_c, the CDF function, $G(.)$, of the limiting null distribution of $\widehat{T}_{n,FRD}$ is continuous and is strictly increasing on $(0,\infty)$, and $G(0) > 1/2$. Then, by the same proof for Theorem 2(b) of Andrews and Shi (2013), it is true that under P_c, $\lim_{\eta\to 0} P(\widehat{T}_{n,FRD} > \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)) = \alpha$ which implies that $\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ h_P\in\mathcal{H}_{cpt}\}} P(\widehat{T}_{n,FRD} > \hat{c}^{\eta,GMS}_{n,FRD}(\alpha)) \geq \alpha$. Then by combining the result in part (b) of the Theorem, we obtain the uniform size control result in part (d).

To show part (c) of the theorem, the asymptotic results in Lemmas B.2, B.3 and B.4 directly imply that $\lim_{\eta\to 0} P^{LFC}_c(\widehat{T}_{n,FRD} > \hat{c}^{\eta,LFC}_{n,FRD}(\alpha)) = \alpha$ under P^{LFC}_c. Then we know $\lim_{\eta\to 0} \limsup_{n\to\infty} \sup_{\{P\in\mathcal{P}_0:\ h_P\in\mathcal{H}_{cpt}\}} P(\widehat{T}_{n,FRD} > \hat{c}^{\eta,LFC}_{n,FRD}(\alpha)) \geq \alpha$. Combining the result in part (a) of the theorem, we obtain the result in part (c). This completes our proof. $\square$

Proof of Theorem 3.2:

Suppose $CLATE(w^*_2, s) > CLATE(w^*_1, s)$ for some $w^*_2 < w^*_1$ and some s. Then by continuity of $CLATE(w,s)$, we can find $w'_2 \ll w'_1$ such that $CLATE(w'_2, s) > CLATE(w'_1, s)$. Again, by continuity of $CLATE(w)$, we can find a small δ such that for all $w''_2 \in \mathcal{N}_{\delta,w}(w'_2)$, $w''_1 \in \mathcal{N}_{\delta,w}(w'_1)$, and $s'' \in \mathcal{N}_{\delta,s}(s)$, we have $w''_2 \ll w''_1$ and

$CLATE(w_2'', s'') > CLATE(w_1'', s'')$. Then we can find a q large enough and $\ddot{\ell} = (\ddot{w}_1, \ddot{w}_2, \ddot{s}, q) \in \mathcal{L}$ such that $\Pi_{j=1}^{d_w}[\ddot{w}_{j1}, \ddot{w}_{j1} + 1/(q+1)] \subseteq \mathcal{N}_{\delta,w}(w_1')$, $\Pi_{j=1}^{d_w}[\ddot{w}_{j2}, \ddot{w}_{j2} + 1/(q+1)] \subseteq \mathcal{N}_{\delta,w}(w_2')$, and $\Pi_{j=1}^{d_s}[\ddot{s}_j, \ddot{s}_j + 1/q] \subseteq \mathcal{N}_{\delta,s}(s)$. It is then straightforward to see that $\mu_{P^*}(\ddot{\ell}) > 0$.

By the definition of $\widehat{T}_{n,FRD}$, we know that $\widehat{T}_{n,FRD} \geq \sqrt{nh}\hat{\mu}_n(\ddot{\ell})/\hat{\sigma}_{\mu,\epsilon}(\ddot{\ell})$, and $\widehat{T}_{n,FRD}$ will diverge to positive infinity when $n \to \infty$, because $\sqrt{nh}\hat{\mu}_n(\ddot{\ell})$ will diverge to positive infinity and $\hat{\sigma}_{\mu,\epsilon}(\ddot{\ell})$ is bounded in probability. Also, both simulated critical values $\hat{c}_{n,FRD}^{\eta,LFC}(\alpha)$ and $\hat{c}_{n,FRD}^{\eta,GMS}(\alpha)$ are bounded in probability. The consistency result of the proposed monotonicity tests then follows. □

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