Can Investing in Hedge Funds Improve Efficiency for Economically Important Investors?

Yu-Chin Hsu, Rachel J. Huang, Larry Y. Tzeng, Christine W. Wang*

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Abstract

The purpose of this paper is to examine the performance of hedge funds from the efficient diversification point of view for economically important investors, which is defined as in Tsetlin et al. (2015). We adopt the generalized almost second-degree stochastic dominance (GASSD) rule proposed by Tsetlin et al. (2015). The rule includes second-degree stochastic dominance as a special case and is a consensus rule for all economically important investors. We establish statistical estimations and tests for the GASSD efficiency of a given portfolio relative to all possible portfolios formed from a given set of assets. We find that for all economically important investors, adding hedge funds to a diversified portfolio can improve efficiency. The results explain the popularity of hedge funds in practice.

JEL classification: D80; D81

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*Yu-Chin Hsu is Associate Research Fellow, Institute of Economics, Academia Sinica, Taiwan. His e-mail address is ychsu@sinica.edu.tw. Rachel J. Huang is Professor, Department of Finance, National Central University, Taiwan. Rachel J. Huang is also Research Fellow at the Risk and Insurance Research Center, College of Commerce, NCCU. Huang can be contacted via e-mail: rachel@ncu.edu.tw. Larry Y. Tzeng is Professor, Department of Finance, National Taiwan University, Taiwan. Larry Y. Tzeng is also Research Fellow at the Risk and Insurance Research Center, College of Commerce, NCCU. His e-mail address is tzeng@ntu.edu.tw. Christine Wang is Research Fellow in the Risk Management Institute at the National University of Singapore. Her e-mail address is rmiww@nus.edu.sg.

1 Introduction

In the last 20 years, hedge funds have become one of the key investment vehicles and sources of capital in the world. Their option-based performance, active trading and diversified strategies have attracted a large amount of money from global investors and also the attention of academia. Several studies document the ability of hedge funds to deliver abnormal returns (Cao et al. 2013; Chen and Liang 2007; Hsu et al. 2014) and also many probe into the causes of their superior performance (Bali et al. 2011, 2012 and 2013; Olmo and Sanso-Navarro, 2012).

Following the literature analyzing the performance of hedge funds, our paper answers the following important question: Can investing in hedge funds improve stochastic dominance efficiency? In the literature, several papers have adopted parameter approaches to evaluate the performance of hedge funds.[1] However, the parameter approaches usually require strong assumptions on either the return distributions or investors' utility functions. Thus, a parameter-free approach such as stochastic dominance is more desirable to characterize the efficiency of hedge funds.

To check whether adding one type of security can improve stochastic dominance efficiency, one commonly used rule is the second-degree stochastic dominance (SSD) criterion.[2] According to Post (2003) and Post and Versijp (2007, PV hereafter), an asset allocation is recognized as an SSD efficient portfolio as long as there is at least one non-satiable ($u' \geq 0$) and risk-averse ($u'' \leq 0$) investor considering it to be an optimal choice.[3] However, the SSD rule is the criterion for all non-satiable and risk-averse investors, including the agents with pathological preferences as pointed out by Leshno and Levy (2002). Thus, a portfolio will be classified as SSD efficient even if only investors with pathological preferences favor it.

[1] For example, Agarwal and Naik (2004) adopted a mean-conditional value-at-risk framework to characterize the systematic risk of hedge funds. Kosowski, Naik, and Teo (2007) used alpha in the seven factor model proposed by Fung and Hsieh (2004) to demonstrate the performance of hedge funds.

[2] The theory of stochastic dominance is developed by Hadar and Russell (1969), Hanoch and Levy (1969), Rothschild and Stiglitz (1970), and Whitmore (1970). To derive tractable tests for stochastic dominance efficiency, Bawa et al. (1985) utilize the convex stochastic dominance condition constructed by Fishburn (1974) to identify the members of optimal and dominated sets when more than two choice alternatives are compared. Post (2003) derives necessary and sufficient tests for the SSD efficiency by considering full diversification across the choice alternatives. He also suggests that these tests could be generalized to third-degree stochastic dominance efficiency. Parallel to Post (2003), Kuosmanen (2004) establishes tests for first-degree stochastic dominance and SSD measures to identify an efficient set with full diversification.

[3] u' and u'' respectively denote the first and the second derivatives of a utility function u.

Let us use an example to demonstrate that most investors could classify one portfolio as an inefficient investment allocation, whereas the SSD rule recognizes it as an efficient one. Assume that, in the feasible set, there are only two exclusive investment outcomes X and Y. X yields 0 with a probability of 10^{-6} and a million dollars with a probability of $1 - 10^{-6}$, and Y yields one dollar with certainty. Note that neither X nor Y dominates the other based on the SSD rule. Thus, both X and Y are considered to be SSD efficient. However, it is obvious that most investors would prefer X to Y. Thus, from most investors' point of view, Y should be classified as a dominated portfolio. In other words, the SSD rule might leave us with too many efficient allocations which include the dominated portfolios from most investors' point of view. Specifically, for the purpose of our paper, even if we find that investing in hedge funds is efficient for risk-averse investors, people might wonder whether only investors with extreme preferences would invest in hedge funds since hedge funds extensively use derivatives, short selling, and leverage.

Thus, instead of SSD rules, we adopt the $(\varepsilon_1, \varepsilon_2)$-generalized almost second-degree stochastic dominance ($(\varepsilon_1, \varepsilon_2)$-GASSD) rules proposed by Tsetlin et al. (2015) to test the efficiency of a given portfolio. Note that Tsetlin et al. (2015) show that $(\varepsilon_1, \varepsilon_2)$-GASSD is a decision criterion to rank distributions for all economically important investors with $u' > 0$, $u'' < 0$ and the ratios of the supremum to infimum of u' and that of $-u''$ are respectively bounded by $\frac{1}{\varepsilon_1} - 1$ and $\frac{1}{\varepsilon_2} - 1$, where both ε_1 and ε_2 are constants between 0 and 0.5. Using the above example, we can show that X dominates Y in terms of $(\varepsilon_1, \varepsilon_2)$-GASSD with reasonable ε_1 and ε_2. Thus, for economically important investors, the $(\varepsilon_1, \varepsilon_2)$-GASSD rule could eliminate many dominated portfolios which are identified as efficient portfolios in terms of SSD.

Our paper is related to Bali et al. (2013) and Denuit et al. (2014). Both of them also employ the concept of almost stochastic dominance to analyze the performance of hedge funds. Bali et al. (2013) employed both ε_1-almost first-degree stochastic dominance and ε_2-almost SSD rules proposed by Leshno and Levy (2002) to evaluate the performance of hedge funds. Our paper differs from theirs in two ways. First, Bali et al. (2013) used pairwise comparison and showed that different types of hedge funds significantly dominate the U.S. equity market and the U.S. Treasury market. We consider full diversification across the choice alternatives rather than pairwise comparisons to examine the efficiency of hedge funds. Second, Bali et al. (2013) adopted the rule of ε_2-almost SSD proposed by Leshno and Levy (2002). Tzeng et al. (2013) showed that the above rule is problematic and re-defined ε_2-almost SSD. Our paper uses the newly-developed $(\varepsilon_1, \varepsilon_2)$-GASSD rules generalized by Tsetlin et al. (2015) which includes the correct ε_2-almost SSD proposed by Tzeng et al. (2013) as a special case.

Denuit et al. (2014) show that the 100% hedge fund portfolio is efficient via their proposed almost marginal conditional stochastic dominance criterion. Our paper is different from theirs in two ways. First, Denuit et al. (2014) only used linear programming to identify an efficient allocation. In our paper, we not only provide an algorithm to identify efficient allocations but also take a step forward to propose a statistical test. Second, the decision criterion is different. The almost marginal conditional stochastic dominance criterion in Denuit et al. (2014) is a rule for investors with $u' > 0$, $u'' < 0$ and confined $-u''$. On the other hand, the $(\varepsilon_1, \varepsilon_2)$-GASSD rule cares about investors with $u' > 0$, $u'' < 0$ and both of their u' and $-u''$ are confined.

In this paper, we introduce a statistical test for the $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency of a given portfolio relative to all possible portfolios formed from investment sets. Our test leverages on the multivariate statistics framework proposed by PV, who develop SSD efficiency tests with superior statistical power properties. We generalize PV's test by adding the preference constraints and derive the corresponding statistics using the optimization method. Without specific assumptions regarding the underlying distribution or functional form specification of utility preferences, our test makes the stochastic dominance rule applicable to any kinds of strategies or investments, especially for hedge fund portfolios.

The data are obtained from the Hedge Fund Research database. After following the screening procedures proposed in the literature, the final sample includes 12,816 hedge funds over the period from January 1994 to December 2011. These hedge funds are further classified into seven broad investment categories: Emerging Markets, Equity Hedge, Event Driven, Fund of Funds (FOF), Macro, Market Neutral, and Relative Value.

To examine whether adding hedge funds can improve efficiency, we first assume that the investment environment only includes the S&P 500 index and the 1-year Treasury Bond. We identify some $(\varepsilon_1, \varepsilon_2)$-GASSD efficient allocations in this two-asset investment environment. We then further examine whether those portfolios become inefficient after adding the third asset: hedge funds. The results show that the efficient portfolios which consist only of stocks and bonds become $(\varepsilon_1, \varepsilon_2)$-GASSD inefficient if Equity Hedge, Event Driven, Macro, Market Neutral, or Relative Value hedge funds are added to the investment universe. The findings suggest that adding the above types of hedge funds could indeed improve efficiency for most economically important investors.

In addition, by assuming that the investment environment includes three assets: hedge funds, stocks and bonds, we find that a 100% hedge fund portfolio cannot be rejected as $(\varepsilon_1, \varepsilon_2)$-GASSD efficient portfolios for all reasonable ε_1's and ε_2's except for Event Driven hedge funds. These results complement the findings of Bali et al. (2013) where they demonstrated that most hedge funds pairwisely dominate the S&P500 index. Our findings show that, from the efficient diversification point of view, most of the 100% hedge fund portfolios are efficient. The empirical evidence also shows that although adding Emerging Market and FOF hedge funds to the investment environment cannot make the efficient portfolios which contain only stocks and bonds become inefficient, a 100% Emerging Market or FOF hedge fund portfolio is an $(\varepsilon_1, \varepsilon_2)$-GASSD efficient portfolio. We further find that although adding Event Driven hedge funds can improve efficiency, a 100% Event Driven hedge fund portfolio is not $(\varepsilon_1, \varepsilon_2)$-GASSD efficient.

Furthermore, by employing Equity Hedge and Market Neutral hedge funds as examples, our empirical evidence shows that many efficient portfolios include positive investment weights on hedge funds. Our empirical results support the findings in reality in that many investors hold hedge funds in their asset allocations. The findings further show that $(\varepsilon_1, \varepsilon_2)$-GASSD rules could substantially reduce the set of efficient portfolios. Specifically, there are $5,145$ SSD efficient portfolios when Equity Hedge is considered. The number of efficient portfolios decreases to 663 when $\varepsilon_1 = 0.059$ and $\varepsilon_2 = 0.022$, which is the threshold suggested by the literature.

2 Generalized Almost Stochastic Dominance

2.1 GASSD and SSD

This section reviews the definition of $(\varepsilon_1, \varepsilon_2)$-GASSD provided in Tsetlin et al. (2015). Let $u(w) : \mathcal{R} \to P$ denote an investor's von Neumann-Morgenstern utility function, where w is the wealth level and P is a nonempty, closed, and convex subset of $\mathcal{R}$. Define the following utility classes:

$$U_2(0,0) = \left\{ u(w) \left| u'(w) \geq 0 \text{ and } u''(w) \leq 0 \right. \right\},$$

$$\text{and } U_2(\varepsilon_1, \varepsilon_2) = \left\{ u \in U_2(0,0) \left| \begin{array}{c} \sup\{u'(w)\} \leq \inf\{u'(w)\}\left(\frac{1}{\varepsilon_1} - 1\right) \\ \sup\{-u''(w)\} \leq \inf\{-u''(w)\}\left(\frac{1}{\varepsilon_2} - 1\right). \end{array} \right. \right\}, \tag{1}$$

where $\varepsilon_i \in \left(0, \frac{1}{2}\right)$, $i = 1, 2$. Thus, $U_2(0,0)$ is the set containing all non-satiable and risk-averse preferences, whereas $U_2(\varepsilon_1, \varepsilon_2)$ is a subset of $U_2(0,0)$ which contains the preferences with confined $u'(w)$ and $u''(w)$. When ε_i decreases, $i = 1, 2$, the set of $U_2(\varepsilon_1, \varepsilon_2)$ becomes larger. If both ε_1 and ε_2 approach zero, then $U_2(\varepsilon_1, \varepsilon_2)$ becomes $U_2(0,0)$.

One example in the set of $U_2(0,0)$ is

$$u(w) = \begin{cases} w \text{ if } w \leq 1 \\ 1 \text{ otherwise.} \end{cases} \tag{2}$$

For the investors with the above utility function, one dollar generates the same utility level as one million dollars. The preference is extreme and pathological. In the example given in the Introduction, an agent with this utility function would prefer Y to X since the expected utility of Y is one and that of X is less than one. To exclude this type of extreme preference, Leshno and Levy (2002) proposed considering economically important agents that have bounded u' and $-u''$. The utility function in (2) is excluded in $U_2(\varepsilon_1, \varepsilon_2)$ for positive ε_1 and/or ε_2.

Let F and G denote two cumulative distribution functions (CDFs). The definitions of SSD and $(\varepsilon_1, \varepsilon_2)$-GASSD are as follows:

Definition 1 *F dominates G in terms of SSD if and only if $E_F(u) \geq E_G(u)$ for all $u \in U_2(0,0)$.*

Definition 2 *F dominates G in terms of $(\varepsilon_1, \varepsilon_2)$-GASSD if and only if $E_F(u) \geq E_G(u)$ for all $u \in U_2(\varepsilon_1, \varepsilon_2)$.*

2.2 GASSD Efficiency

This paper examines $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency in a standard static portfolio choice problem with the following assumptions:[4]

Assumption 1 *Investors with $u \in U_2(\varepsilon_1, \varepsilon_2)$ choose investment portfolios to maximize the expected utility, which is a function of the return on their portfolios. Without loss of generality, we assume that the initial wealth of all investors is one unit.*

Assumption 2 *There are N risky assets and a riskless asset in the investment universe. Investors may diversify between the assets. Let $\boldsymbol{\lambda} \in \mathcal{R}^N$ be a vector of portfolio weights, which could be either positive or negative. If $\boldsymbol{\lambda}^\top \mathbf{1}_T = 1$, then all wealth is invested in risky assets. Let $\boldsymbol{\tau} \in \mathcal{R}^N$ denote the evaluated portfolio.*

Assumption 3 *The excess return vector $\mathbf{x} \in \mathcal{R}^N$ is a random vector which follows a continuous joint CDF $G : R^N \to [0,1]$. R is a nonempty, bounded, and convex subset of $\mathcal{R}$. Let x_f denote the risk-free rate. Thus, the final wealth of investing in $\boldsymbol{\tau}$ is $\mathbf{x}^\top \boldsymbol{\tau} + x_f$. We assume that the mean vector*

$$\alpha(u) = \int u'(\mathbf{x}^\top \boldsymbol{\tau} + x_f)\mathbf{x} dG(\mathbf{x})$$

is finite. Futher assume that the covariance matrix

$$\Omega(u) = \int (u'(\mathbf{x}^\top \boldsymbol{\tau} + x_f)\mathbf{x} - \alpha(u))(u'(\mathbf{x}^\top \boldsymbol{\tau} + x_f)\mathbf{x} - \alpha(u))^\top dG(\mathbf{x})$$

is finite and positive-definite for all $u \in U_2(\varepsilon_1, \varepsilon_2)$.

[4]Note that these assumptions are the same as in PV except that they assume $u \in U_2(0,0)$, whereas we assume $u \in U_2(\varepsilon_1, \varepsilon_2)$.

Based on the above assumptions, the investors' optimization problem is

$$\max_{\boldsymbol{\tau} \in R^N} \int u(\mathbf{x}^\top \boldsymbol{\tau} + x_f) dG(\mathbf{x})$$

and the first-order condition is

$$\alpha(u) = \int u'(\mathbf{x}^\top \boldsymbol{\tau} + x_f) \mathbf{x} dG(\mathbf{x}) = \mathbf{0}_N.$$

Since u is increasing and concave, the second-order condition holds and the above first-order condition is the necessary and sufficient condition for the optimization problem.

We define $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency as follows:

Definition 3 *The evaluated portfolio $\boldsymbol{\tau}$ is $(\varepsilon_1, \varepsilon_2)$-GASSD efficient if and only if $\alpha(u) = \mathbf{0}_N$ for some $u \in U_2(\varepsilon_1, \varepsilon_2)$. Alternatively, $\boldsymbol{\tau}$ is $(\varepsilon_1, \varepsilon_2)$-GASSD inefficient if and only if $\alpha(u) \neq \mathbf{0}_N$ for all $u \in U_2(\varepsilon_1, \varepsilon_2)$.*

When $\alpha(u) = \mathbf{0}_N$ for some $u \in U_2(\varepsilon_1, \varepsilon_2)$, it means that the evaluated portfolio $\boldsymbol{\tau}$ is the optimal choice of these investors. From Definition 2, we know that there is no portfolio dominating $\boldsymbol{\tau}$ via $(\varepsilon_1, \varepsilon_2)$-GASSD. Thus, $\boldsymbol{\tau}$ could be viewed as satisfying $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency. On the other hand, when $\alpha(u) \neq \mathbf{0}_N$ for all $u \in U_2(\varepsilon_1, \varepsilon_2)$, i.e., $\boldsymbol{\tau}$ is not optimal for all investors, it means that there exist other portfolios which can generate higher expected utility than $\boldsymbol{\tau}$. In other words, there are other portfolios that dominate $\boldsymbol{\tau}$ via $(\varepsilon_1, \varepsilon_2)$-GASSD. Thus, $\boldsymbol{\tau}$ is $(\varepsilon_1, \varepsilon_2)$-GASSD inefficient. In addition, from the definition of $U_2(\varepsilon_1, \varepsilon_2)$ in Equation (1), we know that

$$U_2(\varepsilon_1, \varepsilon_2) \subseteq U_2(\theta_1, \theta_2) \text{ for all } \varepsilon_1 \geq \theta_1 \text{ and } \varepsilon_2 \geq \theta_2.$$

Therefore, if the evaluated portfolio $\boldsymbol{\tau}$ is $(\varepsilon_1, \varepsilon_2)$-GASSD efficient, then it is also (θ_1, θ_2)-GASSD efficient.

3 Statistical Estimation and Tests

This section first reviews the multivariate test for SSD efficiency proposed by PV since our test is an extension of theirs. Then, we will formally establish the multivariate test for $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency. To proceed, empirical distributions are employed for the test and the following assumption is imposed:

Assumption 4 *The observations denoted by* $\mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_T)$, *with* $\mathbf{x}_t = (x_{1t}, \ldots, x_{Nt})^\top$ *are serially independently and identically distributed (IID) random draws from the CDF. Note that the timing of the draws is not important. Thus, we label the observations by their ranking with respect to the evaluated portfolio, i.e.,* $\mathbf{x}_1^\top \boldsymbol{\tau} < \mathbf{x}_2^\top \boldsymbol{\tau} < \ldots < \mathbf{x}_T^\top \boldsymbol{\tau}$.

3.1 PV's Multivariate Test

PV constructed the SSD efficiency test based on the first-order conditions for portfolio optimization. Since risk aversion is assumed, the first-order conditions are necessary and sufficient for portfolio optimization. They argued that the evaluated portfolio $\boldsymbol{\tau} \in \mathcal{R}^N$ is SSD efficient if and only if the first-order condition holds for some $u \in U_2(0,0)$.

Let $\beta = (\beta_1, \beta_2, ..., \beta_T)^\top$ denote a gradient vector, $\nabla \mathbf{u}$ of a utility function, u, where $\nabla \mathbf{u} = (u'(\mathbf{x}_1^\top \boldsymbol{\tau} + x_f), u'(\mathbf{x}_2^\top \boldsymbol{\tau} + x_f), ..., u'(\mathbf{x}_T^\top \boldsymbol{\tau} + x_f))^\top$. Let $\mathcal{B}_2(0,0)$ denote a collection of β's such that

$$\begin{aligned} \mathcal{B}_2(0,0) &= \left\{ \nabla \mathbf{u} \,\middle|\, u \in U_2(0,0) \text{ and } T^{-1} \nabla \mathbf{u}^\top \mathbf{1}_T = 1 \right\} \\ &= \left\{ \beta \in R_+^T \,\middle|\, \beta_{t-1} \geq \beta_t,\, t = 2, ..., T \text{ and } \sum_{t=1}^{T} \beta_t = 1 \right\}. \end{aligned} \tag{3}$$

PV impose a standardization in $\mathcal{B}_2(\varepsilon_1, \varepsilon_2)$ so that $\sum_{t=1}^{T} \beta_t = 1$. Imposing this is mainly a computational issue and is without loss of generality. Before we summarize PV's test for the SSD efficiency of an evaluated portfolio $\boldsymbol{\tau} \in \mathcal{R}^N$, let us define more notation. For a given β, define $\hat{\alpha}(\beta)$ and $\widehat{\Omega}(\beta)$ as

$$\hat{\alpha}(\beta) = T^{-1}\mathbf{X}\beta$$
$$\widehat{\Omega}(\beta) = T^{-1}(\mathbf{X} \circ (\mathbf{1}_N\beta^\top) - \hat{\alpha}(\beta)\mathbf{1}_N^T)(\mathbf{X} \circ (\mathbf{1}_N\beta^\top) - \hat{\alpha}(\beta)\mathbf{1}_N^T)^\top$$

PV define the test statistic as

$$J_2 \equiv \min_{\beta \in \mathcal{B}_2(0,0)} T^{-2}\hat{\alpha}(\beta)^\top \widehat{\Omega}^{-1}(\beta)\hat{\alpha}(\beta) \tag{4}$$

We summarize PV's statistical properties of the J_2 in the following theorem:

Theorem 1 ***SSD Test Statistic (PV)*** *Suppose that Assumptions 1-4 hold. Then*

1. *under the null hypothesis that the evaluated portfolio* $\boldsymbol{\tau}$ *is SSD efficient,*

$$\Pr\left[J_2 T > c\,|H_0\right] \leq 1 - \chi^2_N(c)$$

where χ^2_N *denotes the CDF of a Chi-squared distribution with* N *degrees of freedom, and*

2. *under the alternative hypothesis that the evaluated portfolio* $\boldsymbol{\tau}$ *is not SSD efficient,* $\Pr\left[J_2 T > c\,|H_1\right] \to 1$ *for any* $c \in \mathcal{R}$.

The first part of Theorem 1 shows that PV's multivariate test for the SSD efficiency of portfolio $\boldsymbol{\tau}$ can control the size well asymptotically when the critical value is obtained from Chi-squared distributions. The second part of Theorem 1 shows that PV's multivariate test is consistent in that if portfolio $\boldsymbol{\tau}$ is not SSD efficient, we will reject the null hypothesis with a probability approaching one.

3.2 Proposed Test for GASSD Efficiency

We generalize PV's multivariate test to construct the $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency test. To focus on the economically important investors, we consider the utility set $\mathcal{B}_2(\varepsilon_1, \varepsilon_2)$ instead of $\mathcal{B}_2(0, 0)$, where $\mathcal{B}_2(\varepsilon_1, \varepsilon_2)$ denotes a collection of β's such that

$$\begin{aligned}\mathcal{B}_2(\varepsilon_1, \varepsilon_2) &= \left\{ \nabla \mathbf{u} \,\middle|\, u \in U_2(\varepsilon_1, \varepsilon_2) \text{ and } T^{-1}\nabla \mathbf{u}^\top \mathbf{1}_T = 1 \right\} \\ &= \left\{ \beta \in \mathcal{B}_2(0,0) \,\middle|\, \begin{array}{c} \beta_1 \leq \beta_T \left(\frac{1}{\varepsilon_1} - 1\right), \\ \gamma \leq \frac{\beta_{t-1}-\beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}} \leq \gamma \left(\frac{1}{\varepsilon_2} - 1\right), \ t = 2, ..., T \text{ and } \gamma > 0 \end{array} \right\}. \end{aligned} \tag{5}$$

$\mathcal{B}_2(\varepsilon_1, \varepsilon_2)$ is the set of all gradient vectors that are admissible with respect to $U_2(\varepsilon_1, \varepsilon_2)$. To be specific, $\beta \in \mathcal{B}_2(0,0)$ requires that $\beta_{t-1} \geq \beta_t$, $t = 2, ..., T$. Therefore, $\beta_1 \leq \beta_T \left(\frac{1}{\varepsilon_1} - 1\right)$ can be rewritten as

$$\sup\{\beta_t\} \leq \inf\{\beta_t\} \left(\frac{1}{\varepsilon_1} - 1\right)$$

and represents the first condition in the set of $U_2(\varepsilon_1, \varepsilon_2)$ as shown in Equation (1). The variable $\frac{\beta_{t-1}-\beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}}$ represents $-u''$. The condition $\gamma \leq \frac{\beta_{t-1}-\beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}} \leq \gamma \left(\frac{1}{\varepsilon_2} - 1\right)$, $t = 2, ..., T$ and $\gamma > 0$ means that $\gamma \equiv \inf\{\frac{\beta_{t-1}-\beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}}\}$ and

$$\sup\left\{\frac{\beta_{t-1}-\beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}}\right\} \leq \inf\left\{\frac{\beta_{t-1}-\beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}}\right\} \left(\frac{1}{\varepsilon_2} - 1\right),$$

which is the second condition in the set of $U_2(\varepsilon_1, \varepsilon_2)$ as shown in Equation (1).

The test statistic of our test is defined as

$$J_2(\varepsilon_1, \varepsilon_2) \equiv \min_{\beta \in \mathcal{B}_2(\varepsilon_1, \varepsilon_2)} T^{-2} \hat{\alpha}(\beta)^\top \widehat{\Omega}^{-1}(\beta) \hat{\alpha}(\beta) \tag{6}$$

Similar to Theorem 1 for PV's multivariate test, we have the following results for our test.

Theorem 2 *Suppose that Assumptions 1-4 hold. Then*

1. *under the null hypothesis that the evaluated portfolio $\boldsymbol{\tau}$ is $(\varepsilon_1, \varepsilon_2)$-GASSD efficient,*

$$\Pr\left[J_2(\varepsilon_1, \varepsilon_2) T > c \,|H_0\right] \leq 1 - \chi_N^2(c)$$

where χ_N^2 denotes the CDF of a Chi-squared distribution with N degrees of freedom, and

2. *under the alternative hypothesis that the evaluated portfolio* $\boldsymbol{\tau}$ *is not* $(\varepsilon_1, \varepsilon_2)$*-GASSD,* $\Pr[J_2(\varepsilon_1, \varepsilon_2)T > c \,|H_1] \to 1$ *for any* $c \in \mathcal{R}$*.*

Proof. Please see the Appendix. ■

Since $\mathcal{B}_2(\varepsilon_1, \varepsilon_2) \equiv \mathcal{B}_2(0, 0)$ when both ε_1 and ε_2 are zero, our test generalizes PV's test. By varying the preference parameters ε_1 and ε_2, we can test whether adding hedge funds is efficient for all risk-averse investors. We can also test whether adding hedge funds is efficient for economically important investors who belong to the set of $U_2(\varepsilon_1, \varepsilon_2)$.

3.3 Computational Issue

The test statistic $J_2(\varepsilon_1, \varepsilon_2)$ is solved by standard mathematical programming techniques:

$$\begin{aligned} J_2(\varepsilon_1, \varepsilon_2) \quad &\equiv \quad \min_{\beta \in \mathcal{B}_2(\varepsilon_1, \varepsilon_2)} T^{-2}(\mathbf{X}\beta)^\top \widehat{\Omega}^{-1}(\beta)(\mathbf{X}\beta) \\ s.t. \;\; \beta_1 \quad &\leq \quad \beta_T \left(\frac{1}{\varepsilon_1} - 1 \right) \\ \gamma \quad &\leq \quad \frac{\beta_{t-1} - \beta_t}{\mathbf{x}_t^\top \boldsymbol{\tau} - \mathbf{x}_{t-1}^\top \boldsymbol{\tau}} \leq \gamma \left(\frac{1}{\varepsilon_2} - 1 \right), \; t = 2, ..., T \text{ and } \gamma > 0. \end{aligned}$$

Since $\widehat{\Omega}^{-1}(\beta)$ is a complicated function of model variables, PV deal with the problem by using an iterative approach. They give an initial β for a positive definite weighted matrix, and then use this weighted matrix to obtain the optimal solution for β. The procedure is repeated twice in PV to obtain the statistics. Instead of using an iterative approach, we compute the statistics directly. Our approach has the following advantages. First, the conclusion of the test statistic will not be affected by the chosen number of iterations. Second, the quadratic objective function might go to infinity because the weighted matrix generated from the last covariance matrix of returns is singular. Therefore, the iterative approach might reject the efficient portfolio due to numerical issues in some cases. By computing the minimum value, we can avoid this potential bias.

4 Empirical analysis

We proceed with tests to analyze the $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency of the portfolio composed of hedge funds and the U.S. equity and bond markets. We collect data from the Hedge Fund Research database and refer to the literature to eliminate survivorship bias, back-fill bias and multi-period sampling bias. The initial hedge funds contain a total of $11,867$ defunct funds and $6,853$ live funds over the period from January 1994 to December 2011. After the screening procedure, we leave $12,816$ hedge funds in our sample including $7,443$ dead funds and $5,373$ live funds. As in Denuit et al. (2014), these individual hedge funds are grouped into seven broad investment categories: Emerging Markets, Equity Hedge, Event Driven, FOF, Macro, Market Neutral, and Relative Value. The performance of the U.S. equity market and the performance of short-term U.S. Treasury securities are represented by the S&P 500 index returns and the 1-year Treasury Bond returns, respectively.

Table 1 reports the summary statistics of the portfolios over the entire study period under a monthly base. Six out of seven hedge fund investment styles have higher average returns than the S&P 500 index. Among these six hedge funds, five of them have lower standard deviations than the S&P 500 index. They are: Equity Hedge, Event Driven, Macro, Market Neutral, and Relative Value hedge funds. According to the Jarque-Bera (JB) statistics, all of the hedge funds, the S&P 500 index and the 1-year Treasury Bond returns indicate significant departures from normality at the 10% significance level.

[Insert Table 1 here]

4.1 Adding hedge funds

We first answer our main research question by examining whether adding hedge funds to the investment universe can make the efficient portfolios in a world without hedge funds become inefficient. Thus, we first test the efficiency of 11 portfolios that consist of 100% stocks; 90% stocks and 10% bonds; 80% stocks and 20% bonds;...; and 100% bonds by assuming that the investment universe only consists of stocks and bonds.

Regarding the values of ε_1 and ε_2, several values are considered. By conducting experiments with a sample of 200 respondents, Levy et al. (2010) suggest that ε_1 is 0.059.[5] Following their paper, Huang et al. (2015) used a sample of 223, and find that the experimental estimation ε_1 is lower than 0.053 and the estimated ε_2 is 0.022. Thus, we use 0.059 and 0.022 as the benchmarks of ε_1 and ε_2. In addition, we also consider different levels of ε_1 and ε_2 for additional checks. Note that when both ε_1 and ε_2 are 0, $(\varepsilon_1, \varepsilon_2)$-GASSD efficiency reduces to SSD efficiency in PV.

For the sake of brevity, we tabulate only the results with $\varepsilon_1 = 0$, and 0.059, and $\varepsilon_2 = 0$, 0.01, 0.02, 0.022 and 0.03 in Table 2. Panel A of Table 2 reports $J_2(\varepsilon_1, \varepsilon_2)$ for these 11 portfolios. It indicates that all these 11 portfolios are efficient portfolios for all considered values of the ε_1's and ε_2's. In other words, even if the investors with pathological preferences are excluded, these portfolios are still efficient when the investment universe only contains stocks and bonds.

We then relax the assumption and assume that the investment universe consists of three assets: hedge funds, stocks and bonds. Since there are seven types of hedge funds, we examine seven types of investment set: each set includes the S&P500, Treasury Bonds and one hedge fund categorized according to our classification.[6] The efficiency of the above 11 portfolios is examined and the results are presented in the rest Panels of Table 2.

We find that when investors can include either Equity Hedge, Event Driven, Macro, Market Neutral or Relative Value hedge funds in these portfolios, most of these 11 portfolios become inefficient.[7] In other words, adding these hedge funds to the diversified portfolios can indeed improve efficiency. On the other hand, when adding Emerging Market or FOF hedge funds to the investment universe, most of these 11 portfolios are still efficient.

In addition, Table 2 shows that our empirical finding is consistent with the theoretical prediction for $\varepsilon_1 \geq \theta_1$ and $\varepsilon_2 \geq \theta_2$ that: if the null hypothesis that the evaluated portfolio $\boldsymbol{\tau}$ is $(\varepsilon_1, \varepsilon_2)$-GASSD efficient is not rejected, then the null hypothesis that the evaluated portfolio $\boldsymbol{\tau}$ is (θ_1, θ_2)-GASSD efficient is not rejected either.

[Insert Table 2 here]

[5] Levy et al. (2010) also suggest ε_2 as 0.032. However, the decision rule regarding ε_2 adopted in their paper has been corrected by Tzeng et al. (2013). Thus, we do not use 0.032 as the benchmark of ε_2.

[6] Since many markets impose conditions and/or restrictions for short selling strategies, we only consider positive portfolio weights on each asset for simplicity.

[7] Note that the efficient portfolios which contain only stock and bonds are still SSD efficient but not GASSD efficient if Equity Hedge, Market Neutral or Relative Value hedge funds are added to the investment universe.

4.2 The efficiency of 100% hedge fund portfolios

We then examine whether a 100% hedge fund portfolio is $(\varepsilon_1, \varepsilon_2)$-GASSD efficient when the investment universe consists of stocks, hedge funds, and bonds. Panel A in Table 3 presents $J_2(\varepsilon_1, \varepsilon_2)$ for the 100% hedge fund portfolios. It indicates that except for Event Driven hedge funds, a 100% hedge fund portfolio cannot be rejected as $(\varepsilon_1, \varepsilon_2)$-GASSD efficient portfolios for all the reasonable ε_1's and ε_2's. These results complement the findings of Bali et al. (2013), who demonstrated that, for a one-year investment horizon, most hedge funds are dominant assets in terms of ASSD with $\varepsilon_2 = 0.032$ compared to the S&P500 index. Our findings show that, from the efficient diversification point of view, most of the 100% hedge fund portfolios are efficient not only for risk-averse investors but also for economically important risk-averse investors.

It is worth noting that even though adding Emerging Market and FOF hedge funds to the investment environment cannot make the efficient portfolios which contain only stocks and bonds become inefficient as shown in Table 2, Table 3 indicates that a 100% Emerging Market or FOF hedge fund portfolio is an $(0.059, 0.022)$-GASSD efficient portfolio. On the other hand, although adding Event Driven hedge funds can improve efficiency as reported in Table 2, Table 3 shows that a 100% Event Driven hedge fund portfolio is not $(0.059, 0.022)$-GASSD efficient.

Furthermore, Panels B and C in Table 3 present $J_2(\varepsilon_1, \varepsilon_2)$ for the 100% S&P500 portfolios and the 100% bond portfolios, respectively. Panel B of Table 3 indicates that except in the case where Macro hedge funds are considered, the 100% S&P500 portfolio is SSD efficient. However, it is inefficient for most economically important investors except in the case where Emerging Market or FOF hedge funds are included. Panel C demonstrates that when examining the GASSD efficiency, a 100% Treasury Bond portfolio is generally a dominated allocation except that FOF is included in the investment environment.

[Insert Table 3 here]

4.3 The contents of efficient portfolios

Finally, we examine whether there exist many efficient portfolios investing a certain portion in hedge funds when the investment environment includes hedge funds, stocks and bonds. To illustrate the results, we only demonstrate the cases where Equity Hedge or Market Neutral hedge funds are included. The results are shown in Figures 1 and 2, respectively. In each figure, the x-axis denotes the weights of the hedge funds, and the y-axis denotes the weights of the S&P 500 index. The lower triangle area consists of all possible portfolios of these three assets under the constraint that the sum of the investment weights on these three assets is equal to one. We use a grid with step size 0.01 for the portfolio weights. The blue dots are portfolios that passed the $(\varepsilon_1, \varepsilon_2)$-GASSD tests at the 5% significance level, while the remaining portfolios failed the test and are classified as $(\varepsilon_1, \varepsilon_2)$-GASSD inefficient.

Panel A in Figure 1 shows the efficient allocations with $\varepsilon_1 = 0$ by considering Equity Hedge, whereas Panel B shows the efficient allocations with $\varepsilon_1 = 0.059$. These figures indicate that for considered levels of ε_1 and ε_2, the efficient allocations contain positive weights on Equity Hedge. Specifically, when $\varepsilon_1 = 0.059$ and $\varepsilon_2 = 0.022$, the efficient portolios contain at least a 10% holding of Equity Hedge. The results demonstrate that even risk-averse investors without pathological preferences would invest positive weights on hedge funds. These findings partly explain the popularity of hedge funds. In addition, Figure 1 also shows that $(\varepsilon_1, \varepsilon_2)$-GASSD rules could substantially reduce the set of efficient portfolios. Specifically, we find that there are $5,145$ SSD efficient portfolios when Equity Hedge is considered. As shown in Figure 1, when either ε_1 or ε_2 becomes positive, the number of efficient portfolios decreases. For example, when using $\varepsilon_1 = 0.059$ and $\varepsilon_2 = 0.022$, the number of efficient portfolios decreases to 663.

Similar results can be drawn from Figure 2. When setting $\varepsilon_1 = 0.059$ and $\varepsilon_2 = 0.022$, all of the efficient portolios contain at least a 10% holding of Market Neutral hedge funds. Furthermore, $5,139$ portfolios are SSD efficient, whereas only $1,831$ portfolios are $(\varepsilon_1 = 0.059, \varepsilon_2 = 0.022)$-GASSD efficient.

[Insert Figures 1 and 2 here]

5 Conclusion

In this paper, we have established new tests for portfolio allocation by using GASSD criteria. The SSD efficiency test proposed by Post and Versijp (2007) is a special case of ours. By applying the tests, we found that adding hedge funds to a diversified portfolio can improve efficiency. Specifically, we found that when the investment universe includes hedge funds, the efficient portfolios which consist only of stocks and bonds become $(\varepsilon_1, \varepsilon_2)$-GASSD inefficient when certain types of hedge funds are added in the investment universe. Our results have further shown that except for Event Driven hedge funds, 100% hedge fund portfolios are efficient for most risk-averse investors. Furthermore, our empirical evidence has indicated that several efficient portfolios in terms of $(\varepsilon_1, \varepsilon_2)$-GASSD include positive investment weights on hedge funds.

Appendix A   Proof of Theorem 2

For any $u \in U_2(0,0)$, define $\hat{\alpha}(u) = T^{-1}\mathbf{X}\nabla\mathbf{u}$ and $\widehat{\Omega}(u) = T^{-1}(\mathbf{X} \circ (\mathbf{1}_N^\top \nabla\mathbf{u}) - \hat{\alpha}(u)\mathbf{1}_N^T)(\mathbf{X} \circ (\mathbf{1}_N^\top \nabla\mathbf{u}) - \hat{\alpha}(u)\mathbf{1}_N^T)^\top$. It is straightforward to see that the test statistic of our test is equivalent to

$$J_2(\varepsilon_1, \varepsilon_2) \equiv \min_{\beta \in \mathcal{B}_2(\varepsilon_1,\varepsilon_2)} T^{-2}(\mathbf{X}\beta)^\top \widehat{\Omega}^{-1}(\beta)(\mathbf{X}\beta) = \min_{u \in U_2^M(\varepsilon_1,\varepsilon_2)} \hat{\alpha}(u)^\top \widehat{\Omega}^{-1}(u)\hat{\alpha}(u). \tag{A.1}$$

where

$$U_2^M(\varepsilon_1, \varepsilon_2) = \left\{ u \;\middle|\; \begin{array}{c} 0 < u', \; -M < u'' < 0, \\ \sup\{u'\} \le \inf\{u'\}\left(\frac{1}{\varepsilon_1} - 1\right), \;\; \sup\{-u''\} \le \inf\{-u''\}\left(\frac{1}{\varepsilon_2} - 1\right) \\ \text{and the smallest eigenvalue of } \Omega(u) \text{ is bounded away from zero.} \end{array} \right\}$$

Note that imposing $u' < M$ and assuming that the smallest eigenvalue of $\Omega(u)$ is bounded away from zero will not change the problem since such normalization is without loss of generality. The proof for the first part of Theorem 2 is identical to Theorem 2 of PV, so we omit it. To show the second part, note that under the alternative, we have

$$\min_{u \in U_2^M(\varepsilon_1,\varepsilon_2)} \alpha(u)^\top \Omega^{-1}(u)\alpha(u) = \delta > 0.$$

Recall that $\hat{\alpha}(u) = T^{-1}\mathbf{X}\nabla\mathbf{u}$. Note that by the Arzelà-Ascoli Theorem, e.g., Theorem 6.2.61 of Corbae, Stinchcombe and Zeman (2009), $\{\mathbf{x}_t u'(\mathbf{x}_t\boldsymbol{\tau}) | \; u \in U_2^M(\varepsilon_1, \varepsilon_2)\}$ is a compact set with respect to the sup-norm. Therefore, it satisfies Pollard's entropy condition as in Andrews (1994). Then it will satisfy the uniform law of large numbers such that $\sup_{u \in U_2^M(\varepsilon_1,\varepsilon_2)} \|\hat{\alpha}(u) - \alpha(u)\| \xrightarrow{p} 0$. Similarly, $\sup_{u \in U_2^M(\varepsilon_1,\varepsilon_2)} \|\widehat{\Omega}(u) - \Omega(u)\| \xrightarrow{p} 0$. This is sufficient to show that

$$\min_{u \in U_2^M(\varepsilon_1,\varepsilon_2)} \hat{\alpha}(u)^\top \widehat{\Omega}^{-1}(u)\hat{\alpha}(u) \xrightarrow{p} \delta > 0.$$

Therefore, $TJ_2(\varepsilon_1, \varepsilon_2)$ diverges to positive infinity at rate T. This is sufficient to show the second part. ■

References

[1] Agarwal, Vikas and Narayan Y. Naik, 2004, Risks and portfolio decisions involving hedge funds, *Review of Financial Studies* 17, 63-98.

[2] Andrews, D. W. K., 1994, Empirical process methods in econometrics, in: R. F. Engle & D. McFadden (ed.), *Handbook of Econometrics*, volume 4, Chapter 37, 2247-2294, Elsevier.

[3] Bali, Turan G., Stephen J. Brown, and Mustafa Onur Caglayan, 2011, Do hedge funds' exposures to risk factors predict their future returns? *Journal of Financial Economics* 101, 36-68.

[4] Bali, Turan G., Stephen J. Brown, and Mustafa Onur Caglayan, 2012, Systematic risk and the cross section of hedge fund returns, *Journal of Financial Economics* 106, 114–131.

[5] Bali, Turan G., Stephen J. Brown, and K. Ozgur Demirtas, 2013, Do hedge funds outperform stocks and bonds? *Management Science* 59, 1887-1903.

[6] Bawa, Vijay S., James N. Bondurtha Jr., M.R. Rao, and Hira L. Suri, 1985, On determination of the stochastic dominance optimal set, *Journal of Finance* 40, 417-431.

[7] Cao, Charles, Yong Chen, Bing Liang, and Andrew W. Lo, 2013, Can hedge funds time market liquidity? *Journal of Financial Economics* 109, 493-516.

[8] Chen, Yong and Bing Liang, 2007, Do market timing hedge funds time the market? *Journal of Financial and Quantitative Analysis* 42, 827-856.

[9] Corbae, D., M. B. Stinchcombe, and J. Zeman 2009, An introduction to mathematical analysis for economic theory and econometrics, Princeton University Press.

[10] Denuit, Michel M., Rachel J. Huang, Larry Y. Tzeng, and Christine W. Wang, 2014, Almost marginal conditional stochastic dominance, *Journal of Banking and Finance* 41, 57-66.

[11] Fishburn, Peter C., 1974, Convex stochastic dominance with continuous distributions, *Journal of Economic Theory* 7, 143-158.

[12] Fung, W., Hsieh, D., 2004. Hedge fund benchmarks: a risk based approach. *Financial Analyst Journal* 60, 65–80.

[13] Hadar, Josef and William R. Russell, 1969, Rules for ordering uncertain prospects, *American Economic Review* 59, 25-34.

[14] Hanoch, Giora, and Heim Levy, 1969, The efficiency analysis of choices involving risk, *The Review of Economic Studies* 36, 335-346.

[15] Hsu, Yu-Chin, Chung-Ming Kuan, and Meng-Feng Yen, 2014, A generalized stepwise procedure with improved power for multiple inequalities, *Journal of Financial Econometrics* 12, 730-755.

[16] Huang, Rachel J., Yu-Hao Huang, and Larry Y. Tzeng, 2015, Experimental estimation of the preference parameters in almost stochastic dominance, Working paper.

[17] Jean, William H., 1973, More on multidimensional portfolio analysis, *Journal of Financial and Quantitative Analysis* 8, 475-490.

[18] Kosowski, Robert, Narayan Y. Naik, and Melvyn Teo, 2007, Do hedge funds deliver alpha? A Bayesian and bootstrap analysis, *Journal of Financial Economics* 84, 229-264.

[19] Kuosmanen, Timo, 2004, Efficient diversification according to stochastic dominance criteria. *Management Science* 50, 1390-1406.

[20] Leshno, Moshe and Haim Levy, 2002, Preferred by all and preferred by most decision makers: Almost stochastic dominance, *Management Science* 48, 1074-1085.

[21] Levy, Haim, Moshe Leshno, and Boaz Leibovitch, 2010, Economically relevant preferences for all observed epsilon, *Annals of Operations Research* 176, 153-178.

[22] Olmo, Jose and Marcos Sanso-Navarro, 2012, Forecasting the performance of hedge fund styles, *Journal of Banking & Finance* 36, 2351-2365.

[23] Post, Thierry, 2003, Empirical tests for stochastic dominance efficiency, *Journal of Finance* 58, 1905-1931.

[24] Post, Thierry, and Philippe Versijp, 2007, Multivariate tests for stochastic dominance efficiency of a given portfolio, *Journal of Financial and Quantitative Analysis* 42, 489-515.

[25] Rothschild, Michael, and Joseph E. Stiglitz, 1970, Increasing risk: I. A definition, *Journal of Economic Theory* 2, 225-243.

[26] Tsetlin, Ilia, Robert Winkler, Rachel J. Huang, and Larry Y. Tzeng, 2015, Generalized almost stochastic dominance, *Operations Research* 63, 363-377.

[27] Tzeng, Larry Y., Rachel J. Huang, and Pai-Ta Shih, 2013, Revisiting almost second-degree stochastic dominance, *Management Science* 59, 1250-1254.

[28] Whitmore, G.A., 1970, Third-degree stochastic dominance, *The American Economic Review* 60, 457-459.

Table 1: Descriptive Statistics

Asset	Mean	Median	Std Dev	Skewness	Kurtosis	Min	Max	JB	p-value
Emerging	1.08 %	1.74%	4.82	-0.90	7.48	-25.28%	17.79%	202.46	< 0.0001
Equity Hedge	0.89 %	1.12%	2.76	-0.44	5.28	-10.89%	11.06%	51.82	< 0.0001
Event Driven	0.83 %	1.10%	1.86	-1.63	8.74	-8.70%	4.64%	380.72	< 0.0001
FOF	0.50 %	0.62%	1.66	-0.65	6.12	-6.43%	5.95%	99.25	< 0.0001
Macro	0.84 %	0.74%	2.07	0.36	3.07	-4.06%	7.24%	4.78	0.0915
Market Neutral	0.59 %	0.58%	0.87	-0.15	5.73	-3.45%	3.68%	65.21	< 0.0001
Relative Value	0.69 %	0.84%	1.31	-2.90	20.57	-9.06%	4.01%	2971.66	< 0.0001
S&P500 Index	0.56 %	1.12%	4.53	-0.64	3.89	-16.94%	10.77%	21.43	< 0.0001
1-year T-Bond	0.32 %	0.32%	0.29	0.46	3.30	-0.33%	1.31%	8.38	0.0152

This table presents the descriptive statistics of monthly returns on the hedge fund portfolios, S&P500 index, and 1-year Treasury Bond for the sample period from January 1994 to December 2011. We compute the equal-weighted average monthly returns of funds for each of the 7 investment styles. The Jarque-Bera statistic, $JB = n[(S^2/6)+(K-3)^2/24]$, is a formal statistic for testing whether the returns are normally distributed, where n denotes the number of observations, S is skewness and K is kurtosis. JB follows a Chi-square distribution with two degrees of freedom. The last column reports the corresponding p-values.

Table 2: **GASSD Efficiency of portfolio for the S&P500 and Treasury bonds.**

This table displays test statistics and *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels respectively. Panel A considers a chi-squared distribution with 1 degree of freedom, and the others chi-squared distributions with 2 degrees of freedom.

Investment weights		$\varepsilon_1 = 0$					$\varepsilon_1 = 0.059$				
S&P500	vs. Bond	$\varepsilon_2 = 0$	$\varepsilon_2 = 0.01$	$\varepsilon_2 = 0.02$	$\varepsilon_2 = 0.022$	$\varepsilon_2 = 0.03$	$\varepsilon_2 = 0$	$\varepsilon_2 = 0.01$	$\varepsilon_2 = 0.02$	$\varepsilon_2 = 0.022$	$\varepsilon_2 = 0.03$
Panel A: Given an Investment set of the S&P500 and Treasury bonds											
100%	0%	0	0	0	0	0	0	0	0	0	0
90%	10%	0	0	0	0	0	0	0	0	0	0
80%	20%	0	0	0	0	0	0	0	0	0	0
70%	30%	0	0	0	0	0	0	0	0	0	0
60%	40%	0	0	0	0	0	0	0	0	0	0
50%	50%	0	0	0	0	0	0	0	0	0	0
40%	60%	0	0	0	0	0	0	0	0	0	0
30%	70%	0	0	0	0	0	0	0	0	0	0
20%	80%	0	0	0	0	0	0	0	0	0	0
10%	90%	0	0	0	0	0	0	0	0	0	0
0%	100%	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026	0.0026
Panel B: Adding the Emerging Market Hedge Fund to the Investment set											
100%	0%	0.0068	0.0108	0.0135	0.0139	0.0153	0.0068	0.0108	0.0135	0.0139	0.0169
90%	10%	0.0068	0.0109	0.0136	0.014	0.0154	0.0068	0.0109	0.0136	0.014	0.0154
80%	20%	0.0068	0.0111	0.0137	0.0141	0.0155	0.0068	0.0111	0.0137	0.0141	0.0152
70%	30%	0.0068	0.0112	0.0139	0.0143	0.0157	0.0068	0.0112	0.0139	0.0143	0.0163
60%	40%	0.0068	0.0115	0.0141	0.0145	0.016	0.0068	0.0115	0.0141	0.0146	0.0161
50%	50%	0.0064	0.0113	0.0139	0.0142	0.0152	0.0064	0.0114	0.0139	0.0143	0.0192
40%	60%	0.0068	0.0125	0.0148	0.0151	0.0162	0.0068	0.0124	0.0151	0.0155	0.0164
30%	70%	0.0071	0.0135	0.0157	0.0159	0.0171	0.0072	0.0135	0.0156	0.0159	0.017
20%	80%	0.0099	0.0154	0.0175	0.0178	0.0188	0.0099	0.0154	0.0175	0.0178	0.0223*
10%	90%	0.0176	0.0196	0.0205	0.0206	0.0211	0.0176	0.0196	0.0205	0.0206	0.0233*
0%	100%	0.0028	0.0281**	0.0281**	0.0281**	0.0281**	0.0227*	0.0281**	0.0281**	0.0281**	0.0429***

Investment weights		$\varepsilon_1 = 0$					$\varepsilon_1 = 0.059$				
S&P500 vs. Bond		$\varepsilon_2 = 0$	$\varepsilon_2 = 0.01$	$\varepsilon_2 = 0.02$	$\varepsilon_2 = 0.022$	$\varepsilon_2 = 0.03$	$\varepsilon_2 = 0$	$\varepsilon_2 = 0.01$	$\varepsilon_2 = 0.02$	$\varepsilon_2 = 0.022$	$\varepsilon_2 = 0.03$
Panel C: Adding the Equity Hedge Fund to the Investment set											
100%	0%	0.018	0.0237*	0.0296**	0.0517***	0.0537***	0.0264*	0.0369**	0.0438***	0.0452***	0.0537***
90%	10%	0.0149	0.0436***	0.0512***	0.0517***	0.0538***	0.0228*	0.0365**	0.0469***	0.0488***	0.0539***
80%	20%	0.0182	0.024*	0.0302**	0.0315**	0.054***	0.0215*	0.0369**	0.0443***	0.0456***	0.054***
70%	30%	0.0176	0.0233*	0.0304**	0.0317**	0.0543***	0.0187	0.0366**	0.0446***	0.0459***	0.0543***
60%	40%	0.0181	0.0242*	0.0307**	0.032**	0.0546***	0.0184	0.0386**	0.0451***	0.0462***	0.0546***
50%	50%	0.0146	0.0207	0.0515***	0.0521***	0.0545***	0.0205	0.0365**	0.0514***	0.0521***	0.0545***
40%	60%	0.0167	0.0256*	0.032**	0.0537***	0.0556***	0.0185	0.0451***	0.0513***	0.0524***	0.0556***
30%	70%	0.0163	0.0301**	0.0351**	0.0361**	0.0571***	0.0199	0.0376**	0.0512***	0.0521***	0.057***
20%	80%	0.0142	0.0486***	0.0519***	0.0526***	0.0553***	0.0174	0.0502***	0.0533***	0.0539***	0.0559***
10%	90%	0.0153	0.0579***	0.0595***	0.0597***	0.0607***	0.0232*	0.0579***	0.0595***	0.0598***	0.0607***
0%	100%	0.0046	0.0708***	0.0708***	0.0708***	0.0708***	0.0558***	0.0708***	0.0708***	0.0708***	0.0708***
Panel D: Adding the Event Driven Hedge Fund to the Investment set											
100%	0%	0.0251*	0.0546***	0.0601***	0.0613***	0.0638***	0.026*	0.0549***	0.0604***	0.061***	0.0634***
90%	10%	0.0195	0.0368**	0.0606***	0.0612***	0.0638***	0.0218*	0.0417**	0.0606***	0.0612***	0.0638***
80%	20%	0.0194	0.044***	0.0595***	0.0605***	0.0633***	0.0265*	0.0423**	0.0606***	0.0613***	0.0638***
70%	30%	0.0242*	0.0527***	0.0596***	0.0605***	0.0633***	0.0252*	0.0527***	0.0596***	0.0605***	0.0633***
60%	40%	0.0239*	0.0476***	0.0549***	0.0564***	0.0635***	0.0238*	0.0479***	0.0579***	0.0594***	0.0635***
50%	50%	0.0239*	0.0466***	0.0535***	0.0548***	0.0651***	0.0263*	0.0448***	0.0552***	0.0567***	0.0651***
40%	60%	0.0197	0.0526***	0.0584***	0.0606***	0.0636***	0.0281**	0.0508***	0.0582***	0.0591***	0.0624***
30%	70%	0.0213	0.0399**	0.0548***	0.0564***	0.0613***	0.0249*	0.0467***	0.0529***	0.0541***	0.059***
20%	80%	0.0204	0.0403**	0.047***	0.0485***	0.058***	0.0257*	0.0436***	0.0494***	0.0533***	0.0574***
10%	90%	0.0177	0.0376**	0.0427***	0.044***	0.0486***	0.0196	0.0387**	0.0474***	0.051***	0.0604***
0%	100%	0.1117***	0.1117***	0.1117***	0.1117***	0.1117***	0.1117***	0.1117***	0.1117***	0.1117***	0.1117***
Panel E: Adding the FOF Hedge Fund to the Investment set											
100%	0%	0.0024	0.0037	0.0043	0.0044	0.0048	0.0024	0.0035	0.0043	0.0044	0.0048
90%	10%	0.0024	0.0038	0.0043	0.0044	0.0048	0.0024	0.0035	0.0042	0.0044	0.0048
80%	20%	0.0024	0.0035	0.0043	0.0044	0.0048	0.0024	0.0035	0.0043	0.0044	0.0049
70%	30%	0.0024	0.0037	0.0043	0.0045	0.0049	0.0024	0.0035	0.0044	0.0045	0.0049
60%	40%	0.0024	0.0036	0.0044	0.0045	0.005	0.0024	0.0036	0.0044	0.0045	0.005
50%	50%	0.0033	0.0043	0.0054	0.0055	0.006	0.0029	0.0043	0.0053	0.0054	0.006
40%	60%	0.0028	0.0038	0.0047	0.0048	0.0052	0.0025	0.0038	0.0047	0.0048	0.0053
30%	70%	0.0025	0.0041	0.0049	0.005	0.0055	0.0025	0.0041	0.0049	0.005	0.0054
20%	80%	0.0034	0.0048	0.0055	0.0057	0.006	0.0033	0.0047	0.0056	0.0057	0.006
10%	90%	0.0027	0.0065	0.0068	0.0068	0.0071	0.0027	0.0065	0.0068	0.0069	0.0072
0%	100%	0.0114	0.0114	0.0114	0.0114	0.0114	0.0114	0.0114	0.0114	0.0114	0.0114

Investment weights		$\varepsilon_1 = 0$					$\varepsilon_1 = 0.059$				
S&P500 vs. Bond		$\varepsilon_2 = 0$	$\varepsilon_2 = 0.01$	$\varepsilon_2 = 0.02$	$\varepsilon_2 = 0.022$	$\varepsilon_2 = 0.03$	$\varepsilon_2 = 0$	$\varepsilon_2 = 0.01$	$\varepsilon_2 = 0.02$	$\varepsilon_2 = 0.022$	$\varepsilon_2 = 0.03$
Panel F: Adding the Macro Hedge Fund to the Investment set											
100%	0%	0.0553***	0.0575***	0.0581***	0.0582***	0.0586***	0.0554***	0.0574***	0.0581***	0.0582***	0.0586***
90%	10%	0.0555***	0.0575***	0.0583***	0.0584***	0.0586***	0.0553***	0.0575***	0.0583***	0.0584***	0.0586***
80%	20%	0.055***	0.0576***	0.0583***	0.0585***	0.0588***	0.055***	0.0576***	0.0583***	0.0585***	0.0588***
70%	30%	0.055***	0.0579***	0.0584***	0.0585***	0.0587***	0.055***	0.0579***	0.0584***	0.0585***	0.0587***
60%	40%	0.0547***	0.0579***	0.0586***	0.0587***	0.0589***	0.0547***	0.0579***	0.0586***	0.0587***	0.0589***
50%	50%	0.0543***	0.0579***	0.0584***	0.0584***	0.0587***	0.0542***	0.0576***	0.0583***	0.0584***	0.0587***
40%	60%	0.0546***	0.058***	0.0584***	0.0586***	0.0588***	0.0545***	0.058***	0.0584***	0.0586***	0.0588***
30%	70%	0.0555***	0.0579***	0.0584***	0.0584***	0.0587***	0.0558***	0.0576***	0.0581***	0.0582***	0.0585***
20%	80%	0.0552***	0.0579***	0.0584***	0.0585***	0.0588***	0.0552***	0.0579***	0.0584***	0.0585***	0.0588***
10%	90%	0.0509***	0.057***	0.0577***	0.0578***	0.0582***	0.0516***	0.057***	0.0576***	0.0577***	0.0582***
0%	100%	0	0.0664***	0.0673***	0.0677***	0.0679***	0.0053	0.0665***	0.0673***	0.0676***	0.0678***
Panel G: Adding the Market Neutral Hedge Fund to the Investment set											
100%	0%	0.0127	0.0168	0.0261*	0.0312**	0.0393**	0.0229*	0.0456***	0.0518***	0.0528***	0.0561***
90%	10%	0.0123	0.0165	0.0268*	0.0286**	0.0346**	0.0229*	0.0453***	0.0504***	0.0518***	0.0577***
80%	20%	0.0125	0.0169	0.0271*	0.0292**	0.0364**	0.0229*	0.0452***	0.0504***	0.0515***	0.0558***
70%	30%	0.0117	0.0168	0.0287**	0.0307**	0.0374**	0.0229*	0.0451***	0.0506***	0.0518***	0.0559***
60%	40%	0.0144	0.023*	0.0308**	0.0331**	0.0642***	0.0241*	0.0448***	0.0512***	0.0567***	0.0644***
50%	50%	0.012	0.0173	0.0307**	0.0323**	0.0386**	0.0274*	0.0441***	0.0497***	0.0506***	0.055***
40%	60%	0.0078	0.0238*	0.031**	0.0342**	0.04**	0.0323**	0.0439***	0.0498***	0.0509***	0.0549***
30%	70%	0.0068	0.0269*	0.0322**	0.0359**	0.0397**	0.0325**	0.043***	0.0494***	0.0506***	0.0544***
20%	80%	0.0048	0.0286**	0.0333**	0.0342**	0.0394**	0.0323**	0.0427***	0.0492***	0.0503***	0.054***
10%	90%	0.0144	0.0348**	0.0388**	0.0405**	0.0434***	0.0276*	0.0457***	0.0507***	0.0522***	0.0554***
0%	100%	0.1019***	0.1019***	0.1019***	0.1019***	0.1019***	0.1019***	0.1019***	0.1019***	0.1019***	0.1019***
Panel H: Adding the Relative Value Hedge Fund to the Investment set											
100%	0%	0.0143	0.0252*	0.0308**	0.0316**	0.0348**	0.0143	0.0258*	0.0308**	0.0316**	0.0347**
90%	10%	0.0143	0.0258*	0.0307**	0.0314**	0.034**	0.0143	0.0253*	0.0309**	0.0317**	0.0382**
80%	20%	0.0144	0.0257*	0.0307**	0.0315**	0.0342**	0.0142	0.0253*	0.031**	0.0319**	0.0342**
70%	30%	0.0142	0.0254*	0.0309**	0.0317**	0.0343**	0.0141	0.0255*	0.0311**	0.032**	0.0343**
60%	40%	0.0141	0.0257*	0.0311**	0.0318**	0.0345**	0.014	0.0256*	0.0313**	0.0323**	0.0421**
50%	50%	0.0135	0.0254*	0.0307**	0.0316**	0.0343**	0.0135	0.0253*	0.0311**	0.0321**	0.0369**
40%	60%	0.0139	0.0267*	0.0314**	0.0322**	0.0351**	0.0139	0.0264*	0.0318**	0.0327**	0.035**
30%	70%	0.0139	0.0265*	0.0316**	0.0325**	0.0356**	0.0138	0.0264*	0.032**	0.0331**	0.0354**
20%	80%	0.0158	0.0263*	0.0321**	0.0331**	0.0368**	0.0159	0.0262*	0.0322**	0.0332**	0.0361**
10%	90%	0.0101	0.0227*	0.0307**	0.0317**	0.0355**	0.0104	0.024*	0.0296**	0.0308**	0.037**
0%	100%	0.0898***	0.0898***	0.0898***	0.0898***	0.0898***	0.0898***	0.0898***	0.0898***	0.0898***	0.0898***

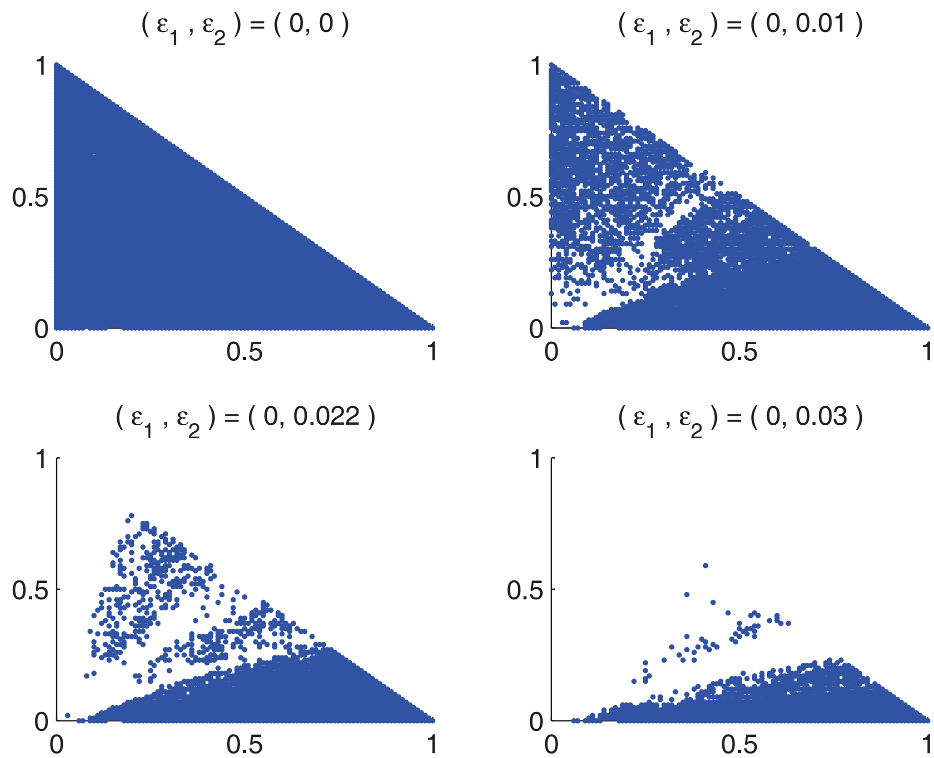
Table 3: **GASSD Efficiency of a Given Asset**

ε_1	ε_2	Emerging Market	Equity Hedge	Event Driven	FOF	Market Macro	Market Neutral	Relative Value
Panel A: 100% hedge fund portfolio								
0.1	0.05	0.0004	0.0217*	0.034**	0	0.0002	0.0038	0.0098
0.1	0.01	0.0001	0.0193	0.0322**	0	0.0001	0.0029	0.0079
0.1	0	0.0001	0.0173	0.0294**	0	0.0001	0.0022	0.004
0.05	0.05	0.0004	0.0202	0.0332**	0	0.0002	0.0037	0.0097
0.05	0.01	0.0002	0.0179	0.0306**	0	0.0001	0.0028	0.0064
0.05	0	0.0001	0.0157	0.0276*	0	0.0001	0.0021	0.004
0	0.05	0.0004	0.0189	0.0331**	0	0.0002	0.0035	0.0097
0	0.01	0.0002	0.0167	0.0301**	0	0.0001	0.0028	0.0066
0	0	0.0001	0.0146	0.0263*	0	0.0001	0.0023	0.0041
0.059	0.022	0.0002	0.0191	0.0315**	0	0.0002	0.0031	0.0086
Panel B: 100% stock portfolio								
0.1	0.05	0.0171	0.057***	0.0685***	0.0055	0.0593***	0.0705***	0.0391**
0.1	0.01	0.0108	0.0456***	0.0459***	0.0036	0.0582***	0.0534***	0.0252*
0.1	0	0.0068	0.0302**	0.0289**	0.0024	0.0554***	0.0296**	0.0161
0.05	0.05	0.0171	0.0571***	0.0706***	0.0055	0.0593***	0.0625***	0.0392**
0.05	0.01	0.0108	0.0352**	0.0558***	0.0036	0.0581***	0.0421**	0.0254*
0.05	0	0.0068	0.0175	0.0254*	0.0024	0.0553***	0.0213	0.0144
0	0.05	0.0171	0.0595***	0.0788***	0.0055	0.0593***	0.0499***	0.0391**
0	0.01	0.0108	0.0354**	0.0453***	0.0035	0.0578***	0.0176	0.0252*
0	0	0.0068	0.0149	0.02	0.0024	0.0553***	0.0118	0.0144
0.059	0.022	0.0139	0.0527***	0.0609***	0.0044	0.0581***	0.0521***	0.0316**
Panel C: 100% bond portfolio								
0.1	0.05	0.0262*	0.0669***	0.1098***	0.0101	0.0624***	0.1022***	0.0895***
0.1	0.01	0.0262*	0.0669***	0.1098***	0.0101	0.0618***	0.1022***	0.0895***
0.1	0	0.0212	0.0527***	0.1098***	0.0101	0.0174	0.1022***	0.0895***
0.05	0.05	0.0262*	0.0736***	0.1098***	0.0101	0.0375**	0.1022***	0.0895***
0.05	0.01	0.0262*	0.0709***	0.1098***	0.0101	0.0263*	0.1022***	0.0895***
0.05	0	0.0214*	0.0533***	0.1098***	0.0101	0.003	0.1022***	0.0895***
0	0.05	0.0262*	0.0727***	0.1098***	0.0101	0.0273*	0.1022***	0.0895***
0	0.01	0.0262*	0.0706***	0.1098***	0.0101	0.0032	0.1022***	0.0895***
0	0	0.0023	0	0.1098***	0.0101	0.0001	0.1022***	0.0895***
0.059	0.022	0.0281**	0.0708***	0.1117***	0.0114	0.0679***	0.1019***	0.0898***

This table displays the test statistic, J_2, and *, **, and *** denote statistical significance at the 10%, 5%, and 1% levels of $\Pr\left[J_2(\varepsilon_1,\varepsilon_2)T > \chi^2_N(1-\alpha)\,|H_0\right] \leq 1-\alpha$ with $N = 2$ degrees of freedom.

Figure 1: Equity Hedge

(a) $\epsilon_1 = 0$



(b) $\epsilon_1 = 0.059$

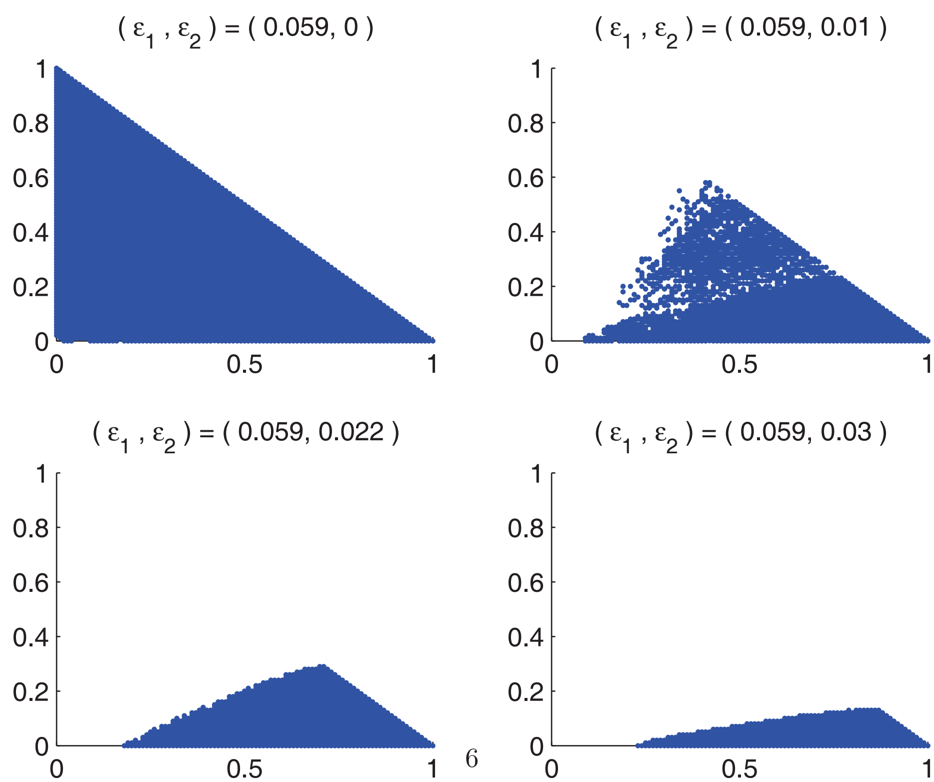
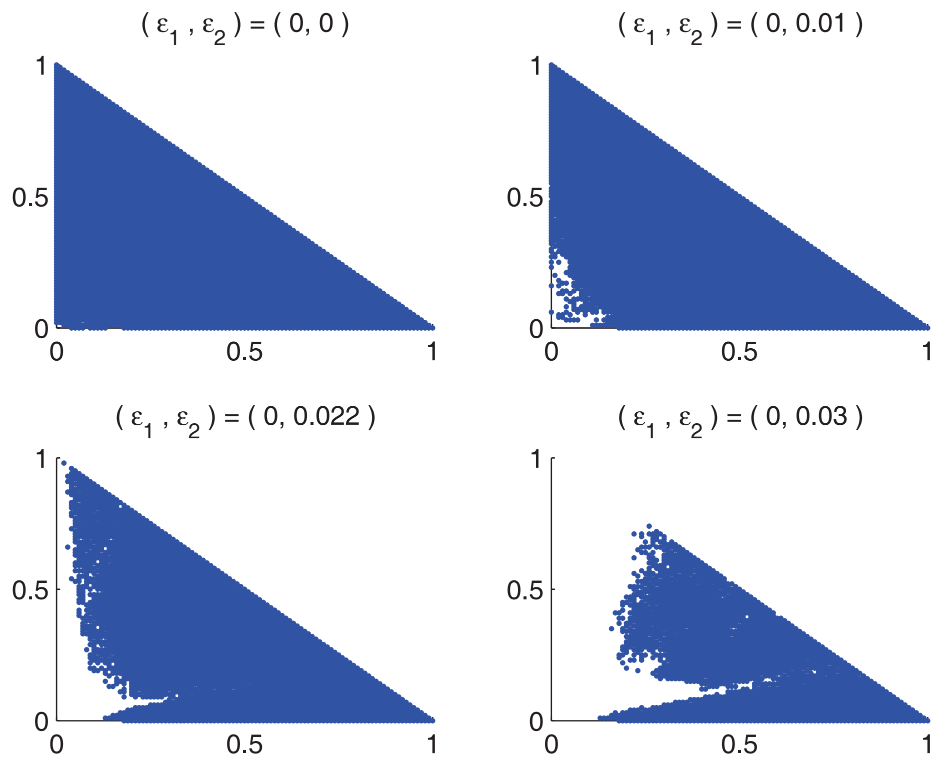


Figure 2: Market Neutral

(a) $\epsilon_1 = 0$



(b) $\epsilon_1 = 0.059$